Transformations and alignment

Outline

- Review from last lec
- Transformations (2D/3D)
- Direct methods
- Lucas Kanade

Review from last lecture

$$\lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} fs_x & fs_\theta & o_x \\ 0 & fs_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$= K_{3\times 3} \begin{bmatrix} R_{3\times 3} & T_{3\times 1} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

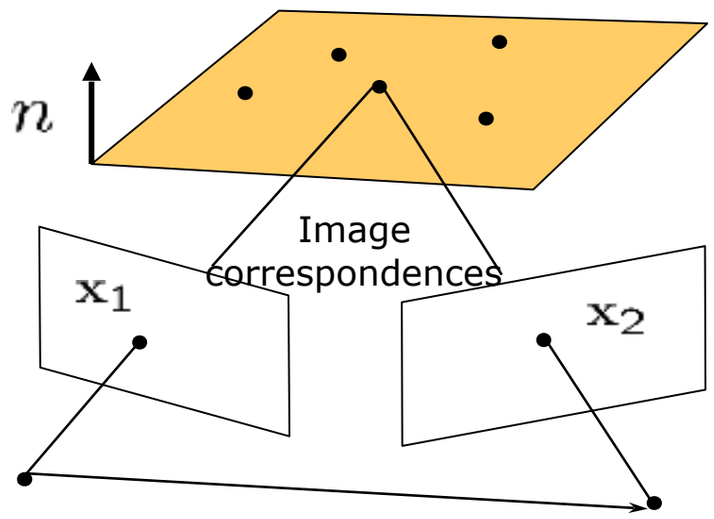
$$= M_{3\times 4} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Claims:

1. A 3x4 matrix ‘M’ can be a camera matrix iff det(A) is not zero (allows one to cast a ray)
2. M is determined only up to a scale factor (easy to show)

Computing homography projections

Given (x_1,y_1) and H, how do we compute (x_2,y_2)?

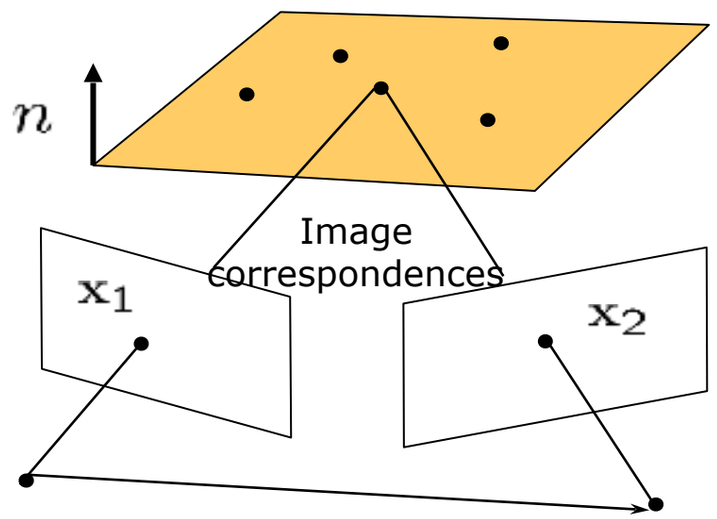


$$\lambda \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

$$x_2 = \frac{\lambda x_2}{\lambda} = \frac{ax_1 + by_1 + c}{gx_1 + hy_1 + i}$$

Estimating homographies

Given corresponding 2D points in left and right image, estimate H



$$x_2(gx_1 + hy_1 + i) = ax_1 + by_1 + c$$

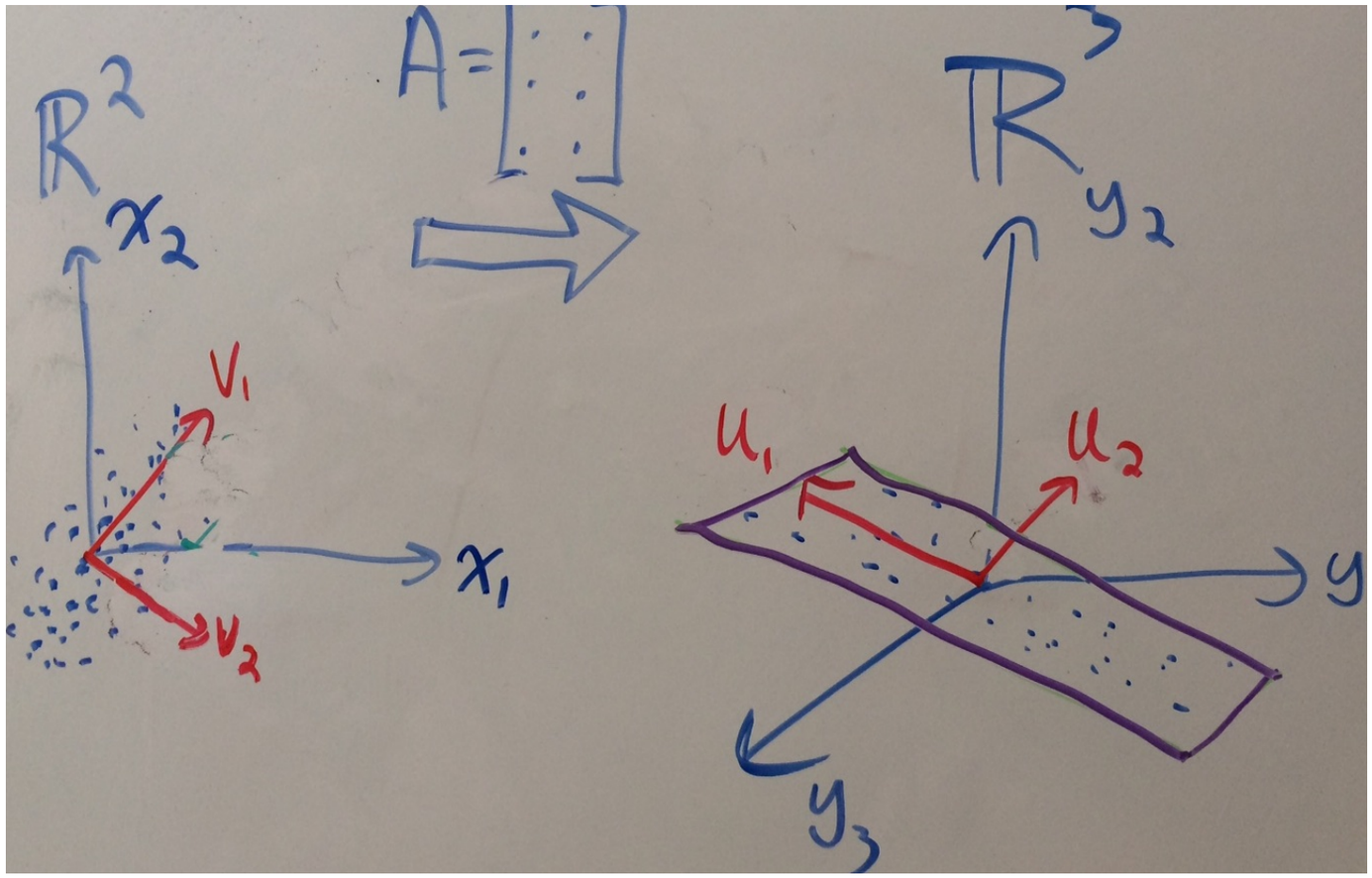
$$\vdots$$

$$AH(:) = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}$$ Homogenous linear system

$$\min_{||H(:)||^2=1} ||AH(:)||^2$$

Recall: SVD

$$y = U\Sigma V^T x$$



Any linear operator can be thought of as mapping from R^n to R^m

1. projection (with right singular vectors in V)
2. scaling (with singular values in Sigma),
3. reconstruction (with left singular vectors in U)

Homogenous least-squares: $\min_{h:h^T h=1} ||Ah||^2 = V(:, end)$

RANSAC for estimating transformation

RANSAC loop:

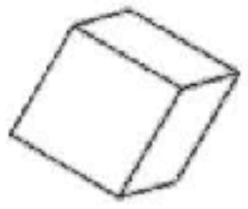
1. Select feature pairs (at random)
2. Compute transformation T (exact)
3. Compute *inliers* (point matches where $|p_i' - \boldsymbol{T}\, p_i|^2 < \varepsilon$)
4. Keep largest set of inliers

Because transformation is fit using minimal # of points,
algebraic (homogenous least squares) solution often suffices
(e.g., noise-free soln exists when estimating H from 4 pts)

Outline

- Review from last lec
- Transformations (2D/3D)
- Direct methods
- Lucas Kanade

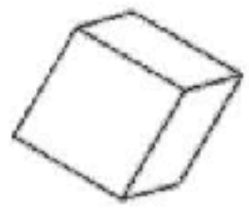
Transformations in 3D

Similarity 7dof	$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$		Preserves angles, ratios of length
Euclidean 6dof	$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$		Preserves angles, lengths

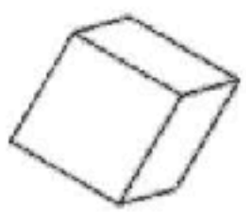
Make use of 3D homogenous coordinates

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Transformations in 3D

Affine 12dof	$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$		Preserves parallelism, volume ratios
Similarity 7dof	$\begin{bmatrix} sR & t \\ 0^- & 1 \end{bmatrix}$		Preserves angles, ratios of length
Euclidean 6dof	$\begin{bmatrix} R & \mathbf{t} \\ 0^- & 1 \end{bmatrix}$		Preserves angles, lengths

Transformations in 3D

Projective 15dof	$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$		Preserves intersection and tangency
Affine 12dof	$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$		Preserves parallelism, volume ratios
Similarity 7dof	$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$		Preserves angles, ratios of length
Euclidean 6dof	$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$		Preserves angles, lengths

Normalize by last coordinate to recover 3D points

$$\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \\ \lambda \end{bmatrix}$$

Transformations in 2D

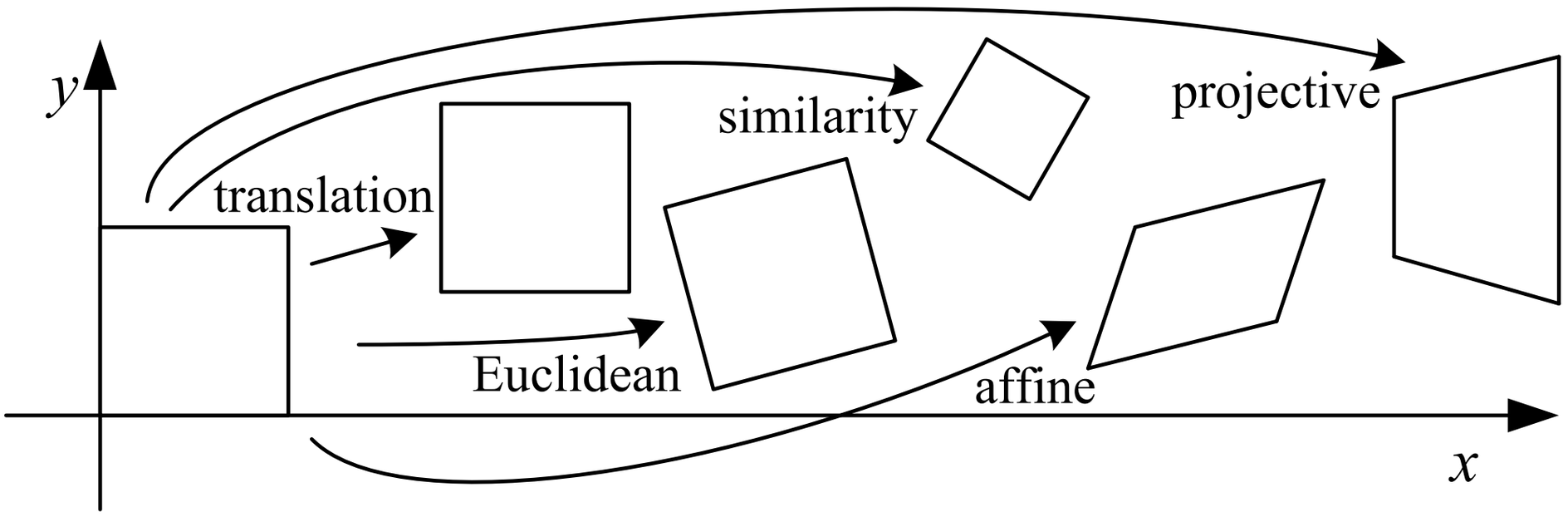
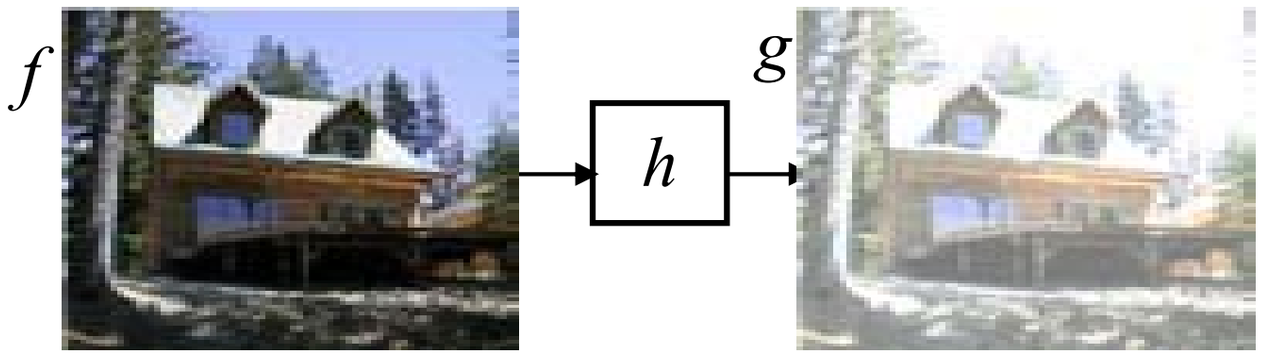
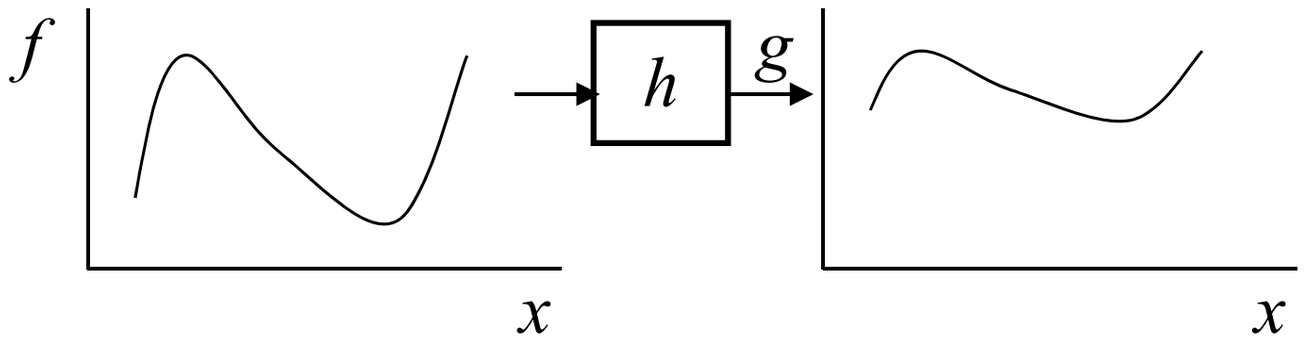


Image warping

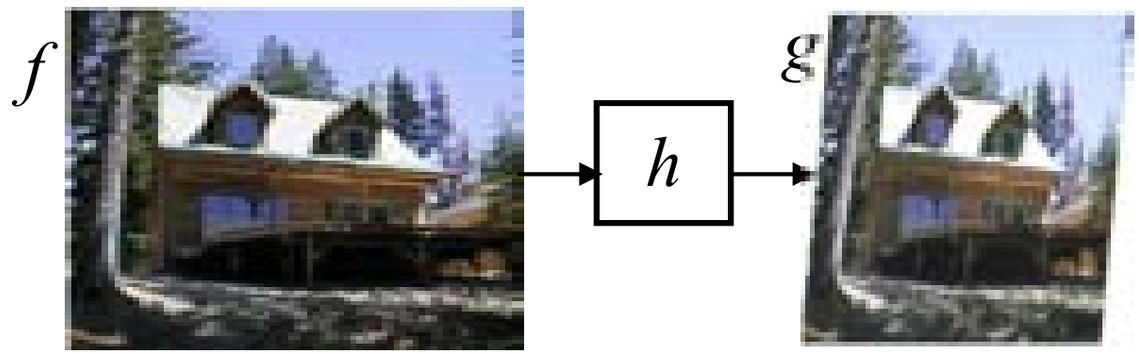
Filtering

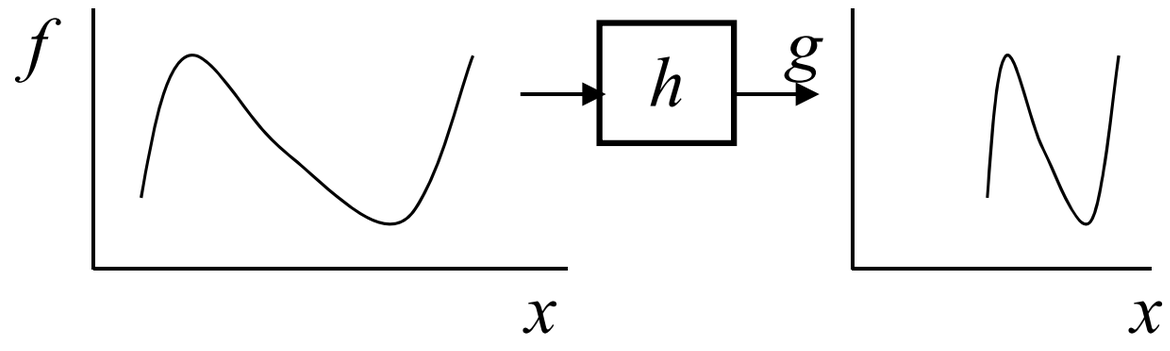




$$g(\mathbf{x}) = h(\mathbf{x}) * f(\mathbf{x})$$

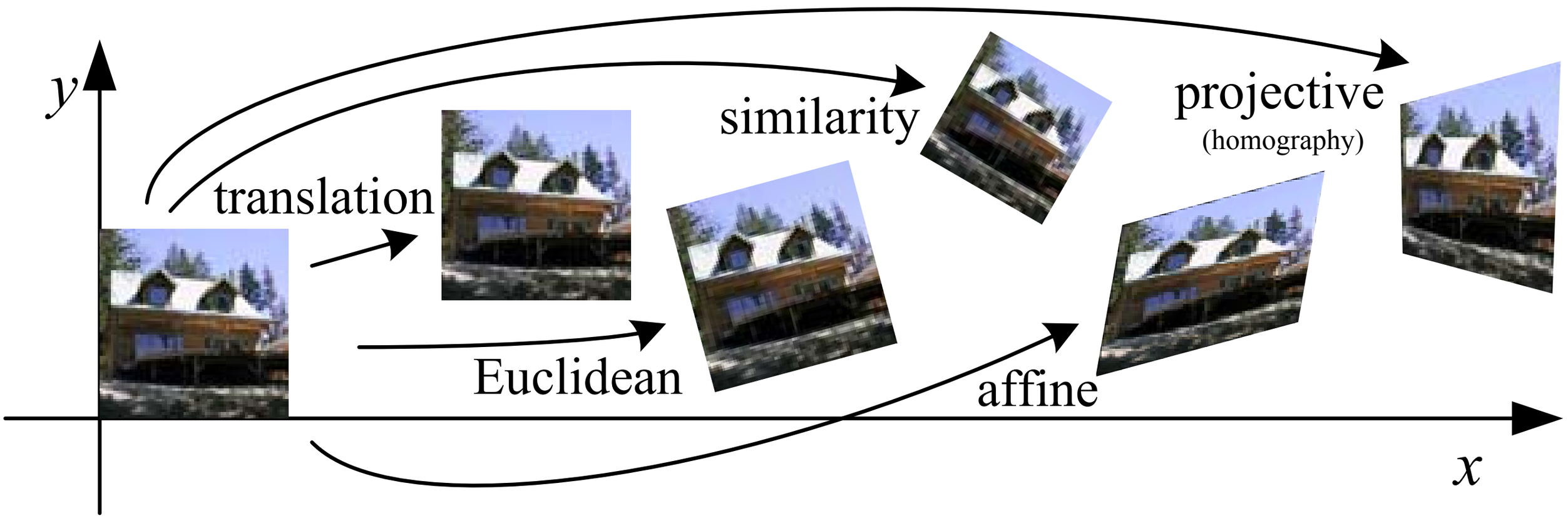
Warping



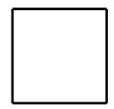


$$g(\mathbf{x}) = f(h(\mathbf{x}))$$

Family of image warps



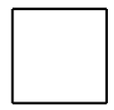
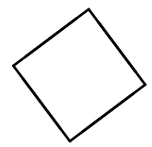
Family of 2D warps

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	2	orientation	

$$x' = x + t_x$$
$$y' = y + t_y$$

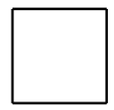
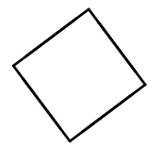
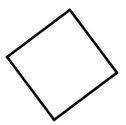
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Family of 2D warps

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\left[\begin{array}{c\|c} \boldsymbol{I} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\left[\begin{array}{c\|c} \boldsymbol{R} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	3	lengths	

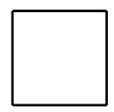
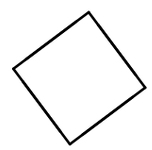
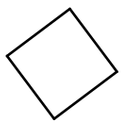
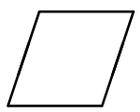
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Family of 2D warps

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\left[\begin{array}{c\|c} \boldsymbol{I} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\left[\begin{array}{c\|c} \boldsymbol{R} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	3	lengths	
similarity	$\left[\begin{array}{c\|c} s\boldsymbol{R} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	4	angles	

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s\cos\theta & -\sin\theta & t_x \\ \sin\theta & s\cos\theta & t_y \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Family of 2D warps

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\left[\begin{array}{c\|c} \boldsymbol{I} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\left[\begin{array}{c\|c} \boldsymbol{R} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	3	lengths	
similarity	$\left[\begin{array}{c\|c} s\boldsymbol{R} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	4	angles	
affine	$\left[\begin{array}{c} \boldsymbol{A} \end{array}\right]_{2\times 3}$	6	parallelism	

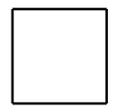
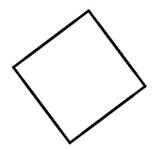
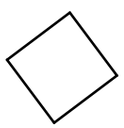
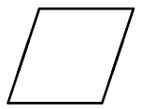
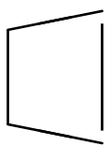
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Relates the image projections of the same 3D scene under 2 affine cameras (will prove in future lectures)

Think of as change of basis and 2D translation

How is this different than a linear transformation?

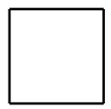
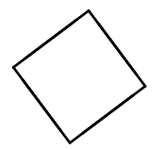
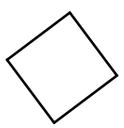
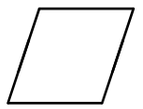
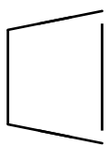
Family of 2D warps

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\left[\begin{array}{c\|c} \boldsymbol{I} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\left[\begin{array}{c\|c} \boldsymbol{R} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	3	lengths	
similarity	$\left[\begin{array}{c\|c} s\boldsymbol{R} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	4	angles	
affine	$\left[\begin{array}{c} \boldsymbol{A} \end{array}\right]_{2\times 3}$	6	parallelism	
projective	$\left[\begin{array}{c} \tilde{\boldsymbol{H}} \end{array}\right]_{3\times 3}$	8	straight lines	

$$\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

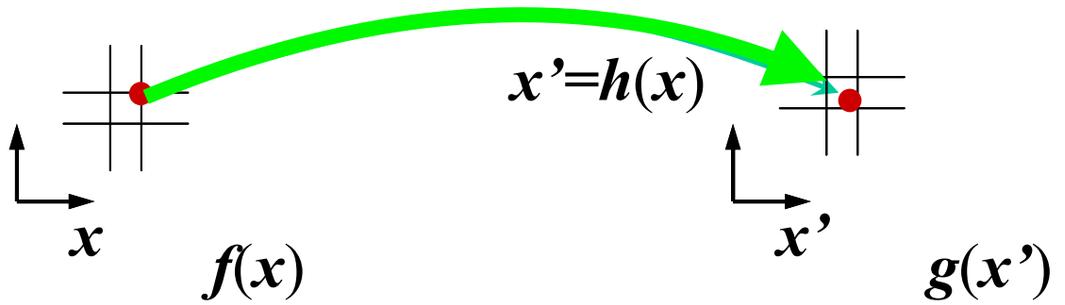
Relates the image projections of
(1) planar scene under pinhole cameras or (2) any scene under rotated cameras

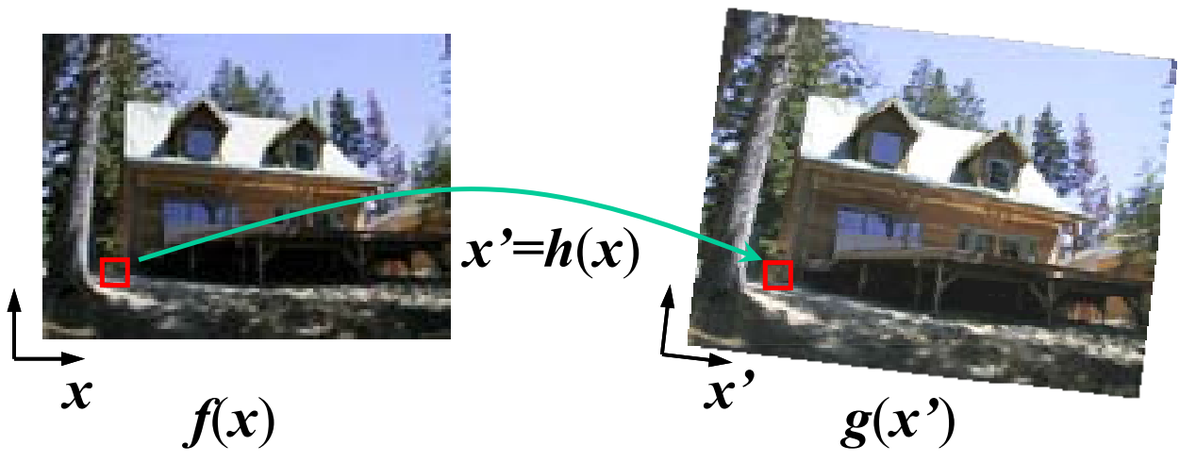
Family of 2D warps

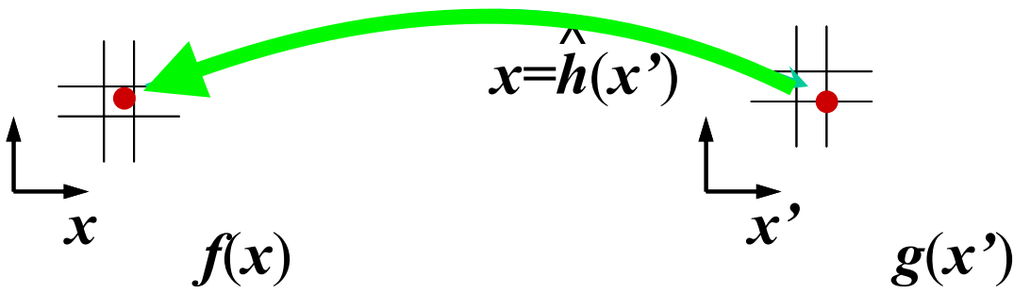
Transformation	Matrix	# DoF	Preserves	Icon
translation	$\left[\begin{array}{c\|c} \boldsymbol{I} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\left[\begin{array}{c\|c} \boldsymbol{R} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	3	lengths	
similarity	$\left[\begin{array}{c\|c} s\boldsymbol{R} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	4	angles	
affine	$\left[\begin{array}{c} \boldsymbol{A} \end{array}\right]_{2\times 3}$	6	parallelism	
projective	$\left[\begin{array}{c} \tilde{\boldsymbol{H}} \end{array}\right]_{3\times 3}$	8	straight lines	

Do Euclidean transformations relate image projections of same scene under orthographic cameras?

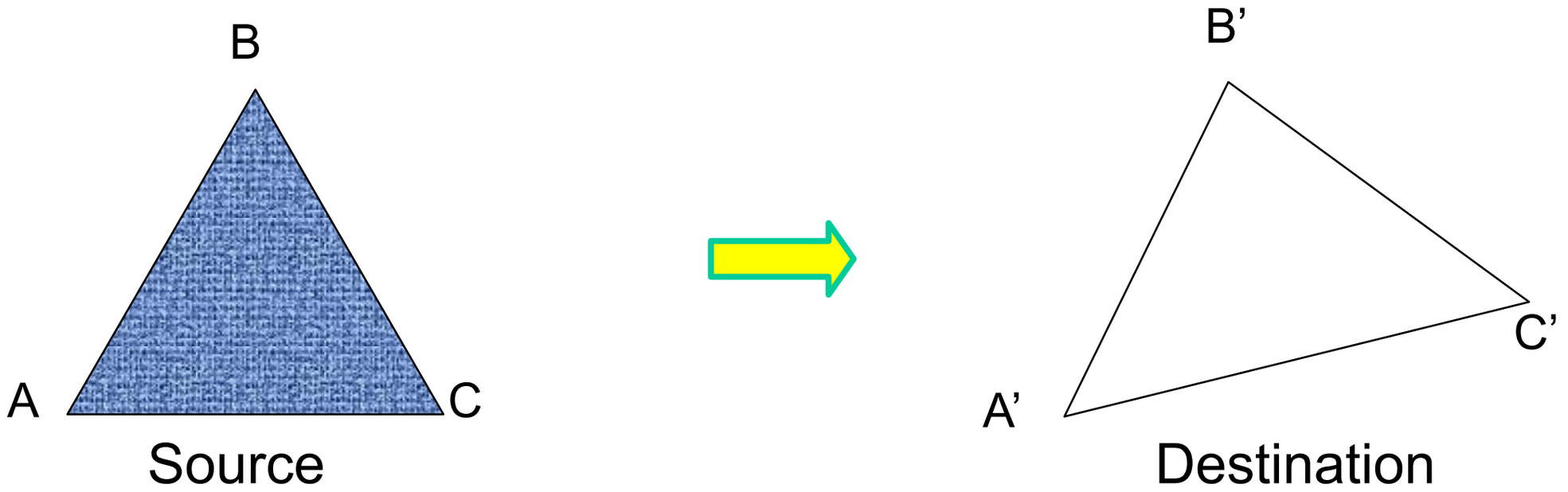
Forward vs inverse warping







Example: warping triangles



What kind of transformation is this? How many DOFs?

Example application: image morphing

Piecewise affine warps (cut each quadrilateral into 2 triangles)

Example application: shape modeling

D'Arcy Thompson
"On Growth and Form" 1915

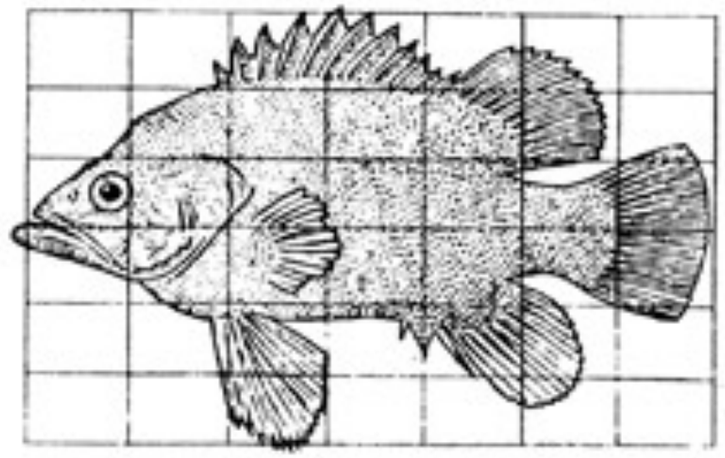


Fig. 150. *Polyprion.*

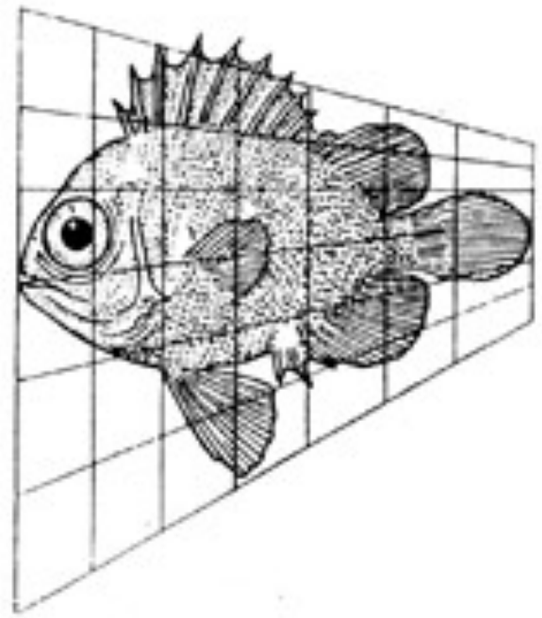


Fig. 151. *Pseudopriacanthus altus.*

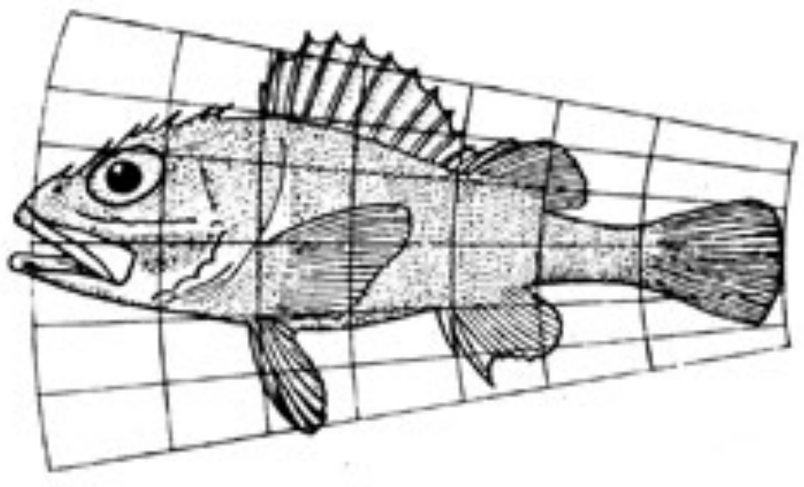


Fig. 152. *Scorpaena* sp.

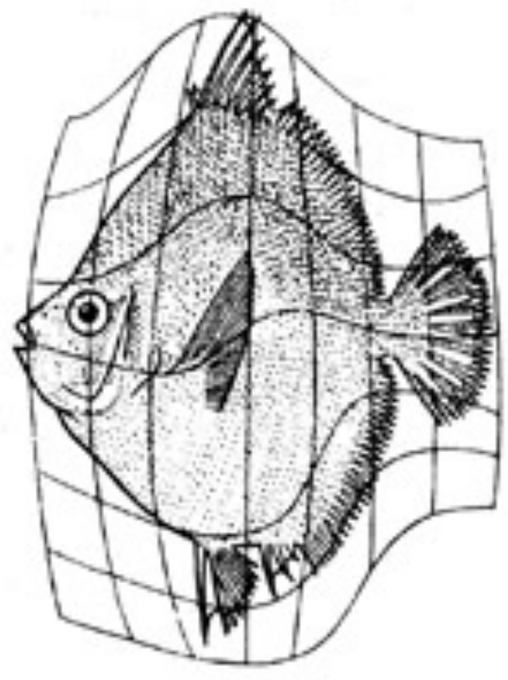
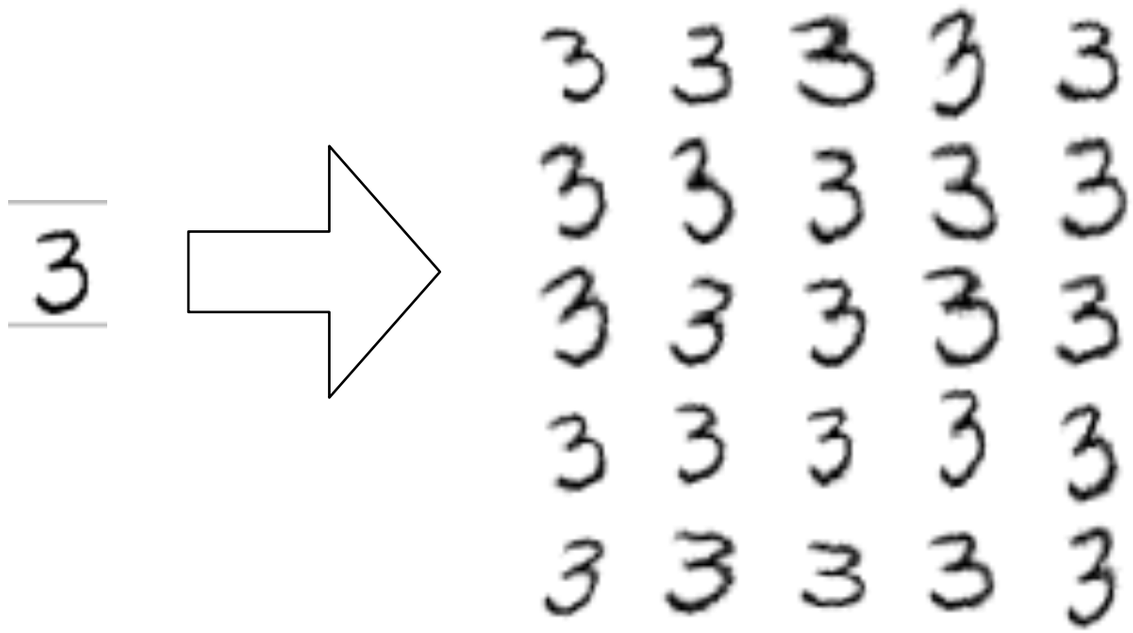


Fig. 153. *Antigonia capros.*

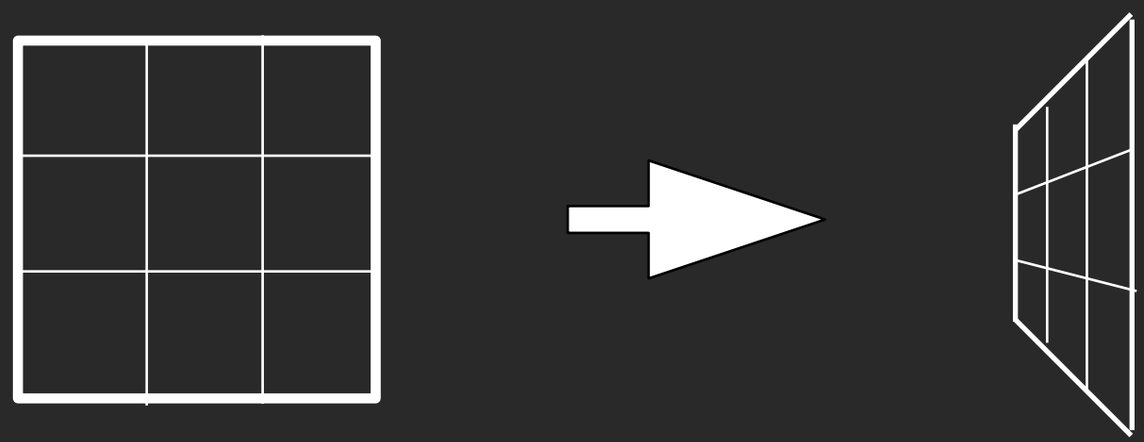
Example application: data augmentation



Hypothesis: we'll see more of this in the future!

Data augmentation by geometric warping

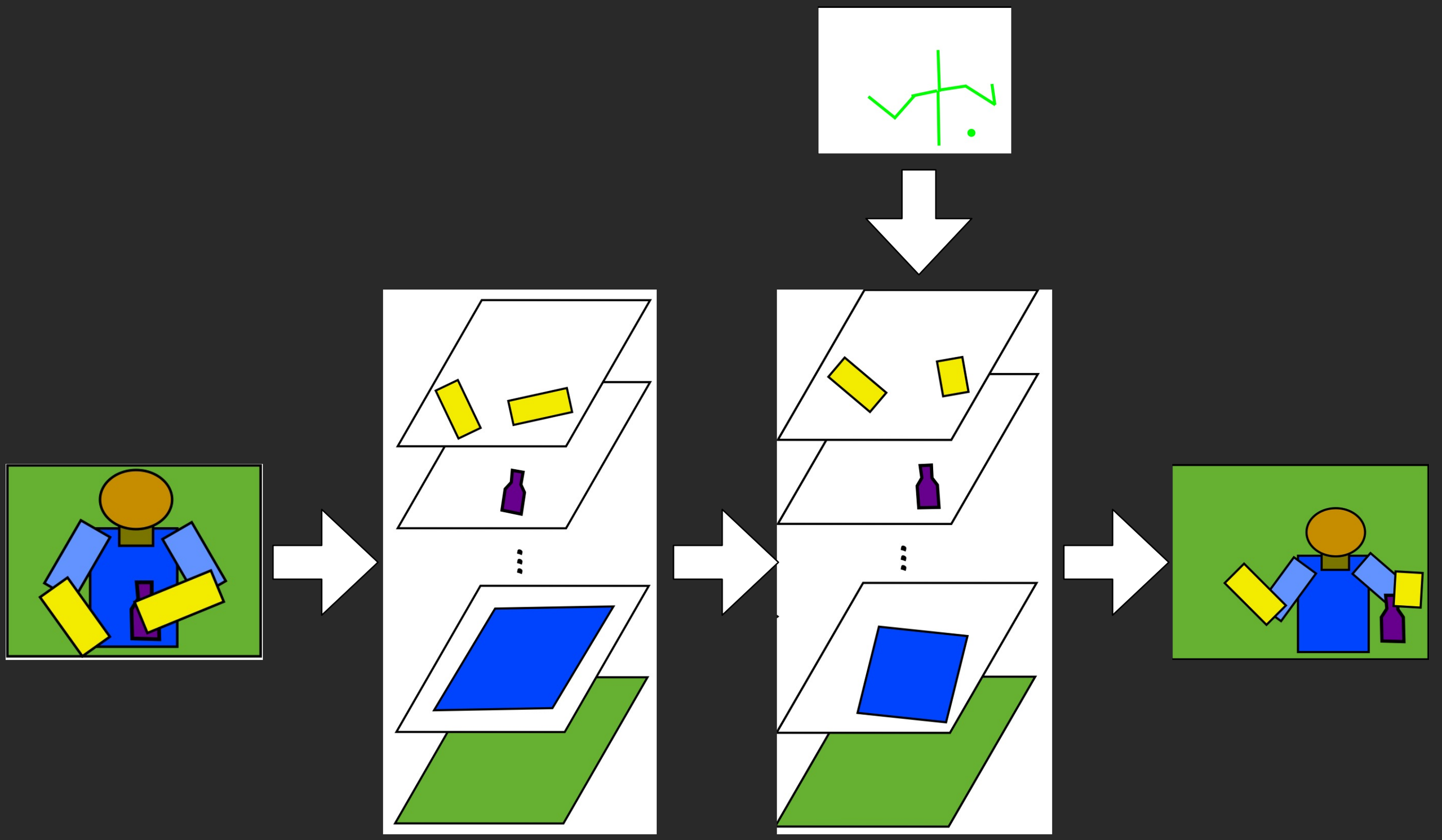
Cabinet Cargo Container Dryer Cardboard Box



Example application: interactive video tracking

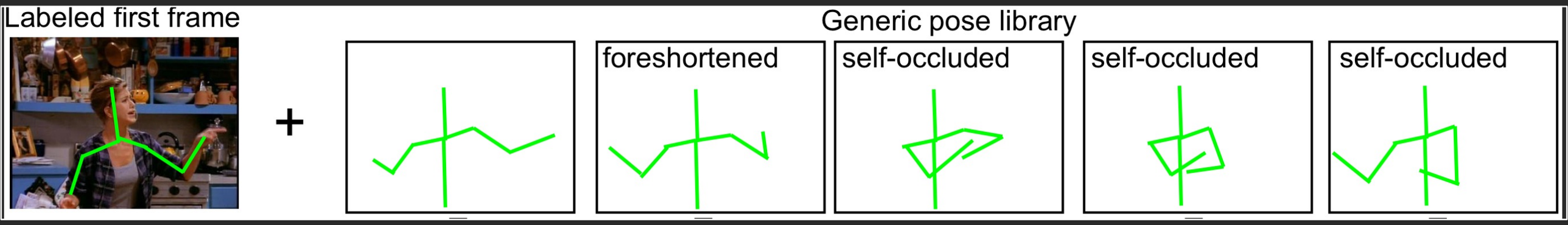
Labeled first frame

Layered (2.1D) warps



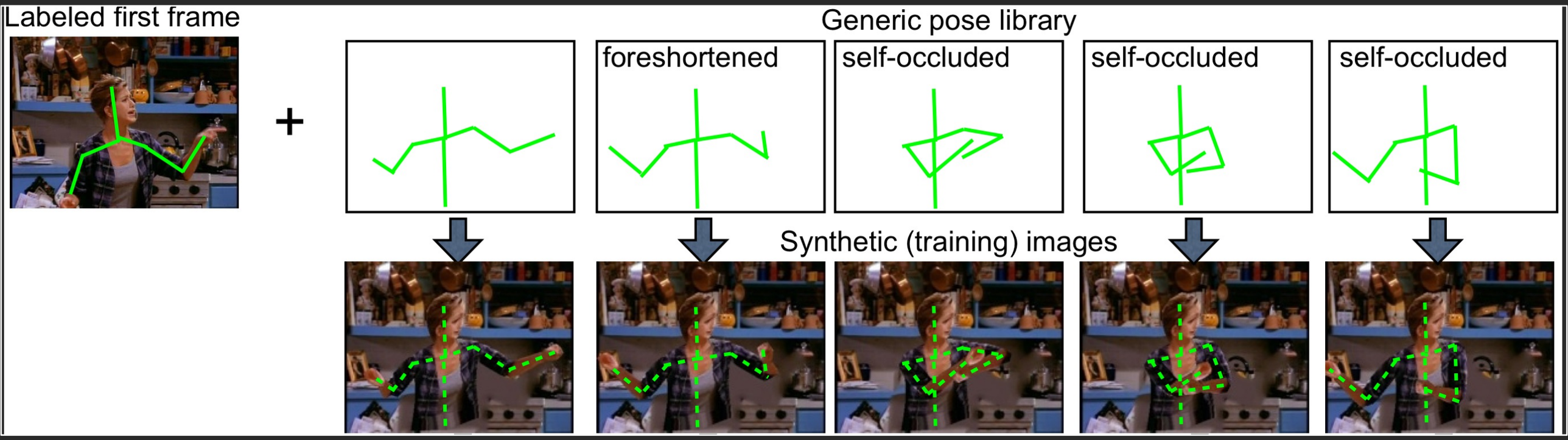
2D similarity transformations

Overall pipeline

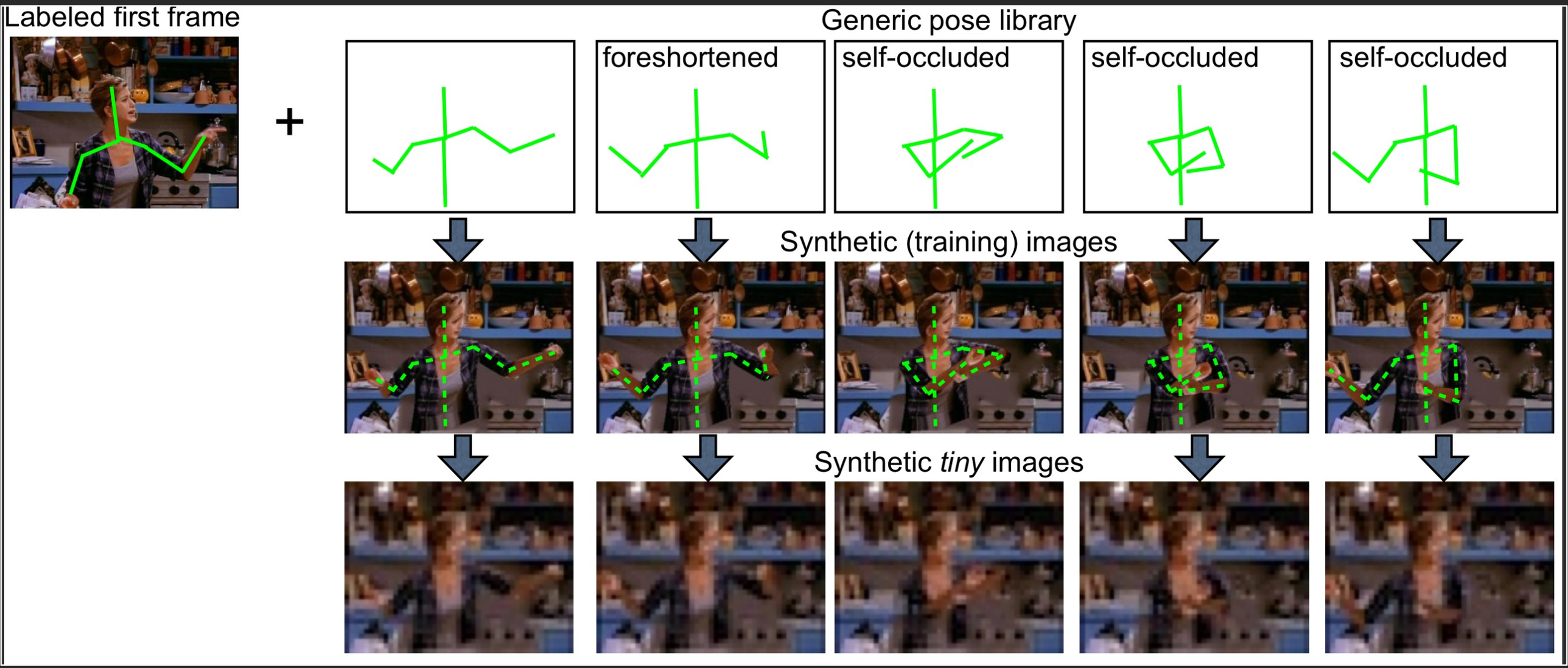


(lots of domain knowledge)

Overall pipeline



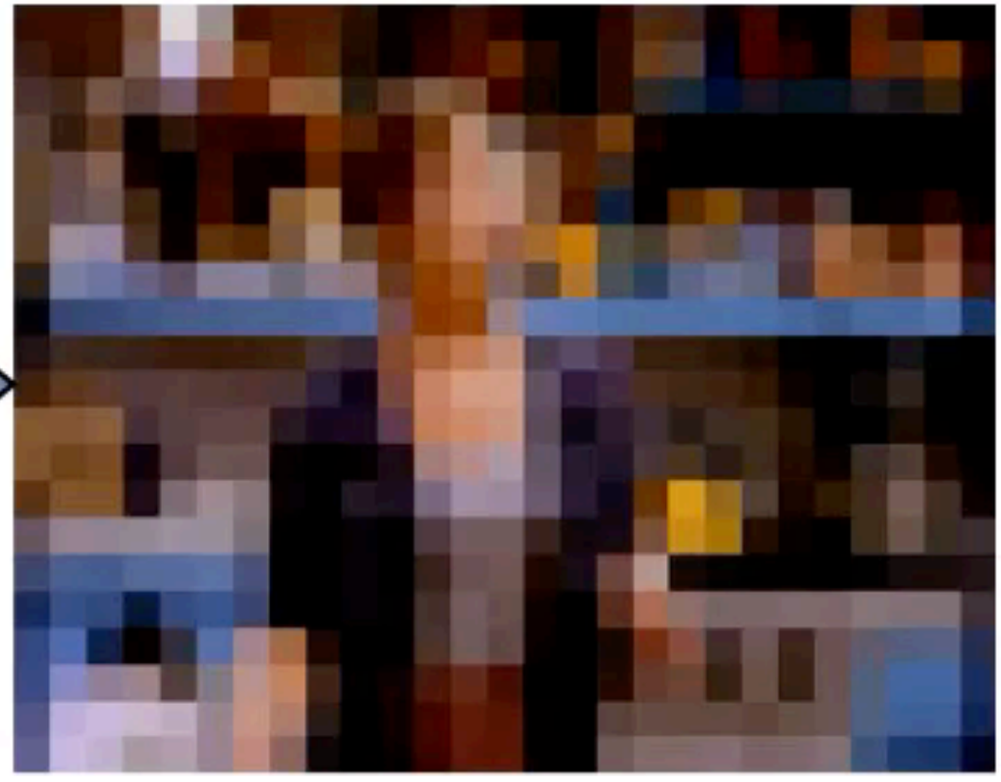
Overall pipeline



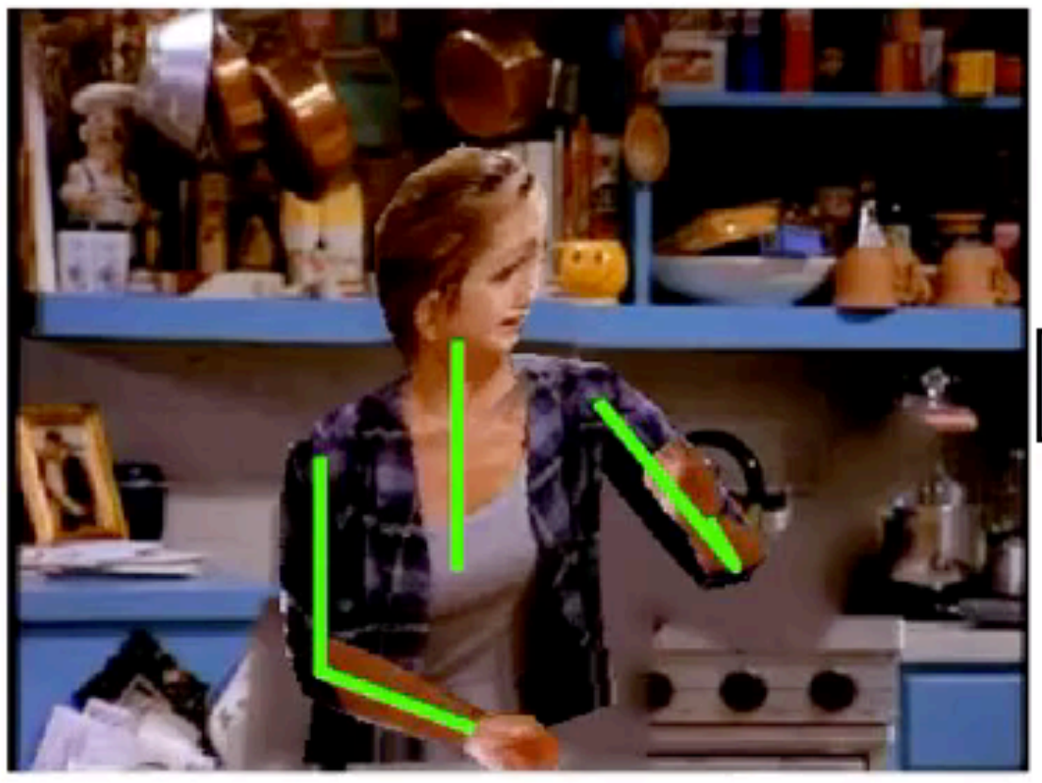
original

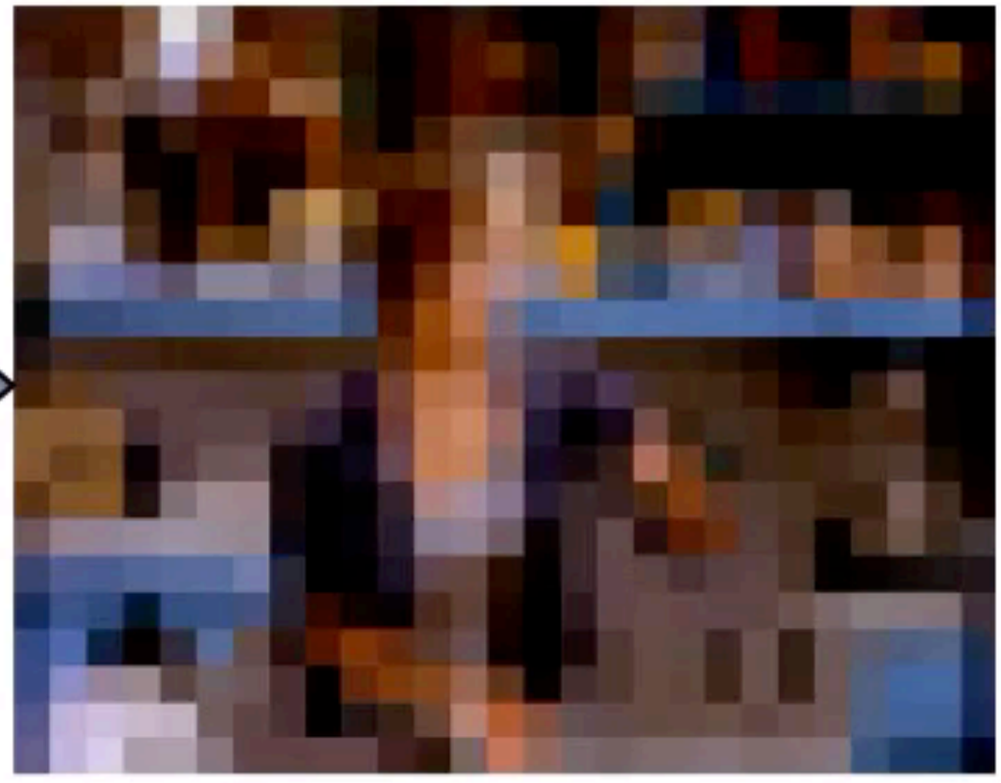
low resolution

test video



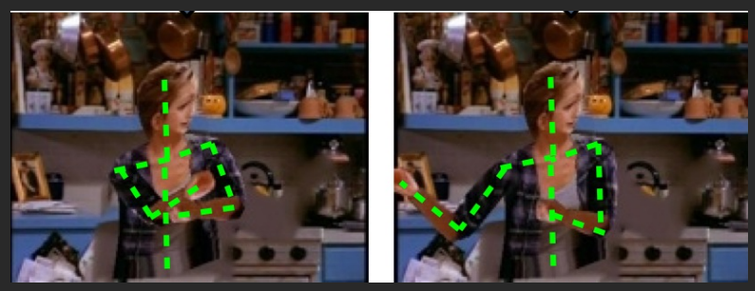
best match



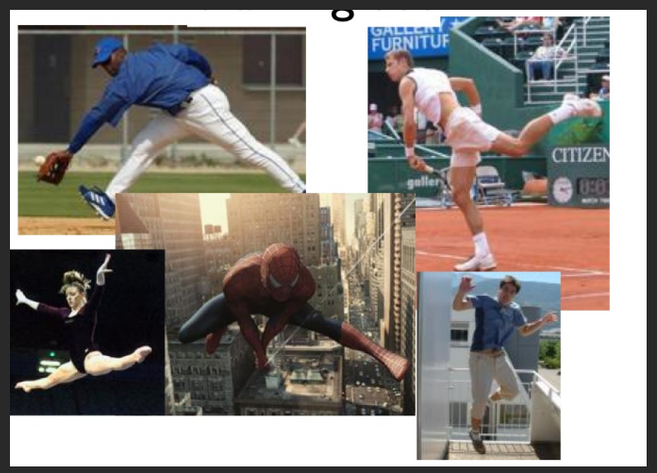


Some observations

1. Layered pixel models are particularly effective for occlusions



2. We do not need to synthesis appearance variations for a video

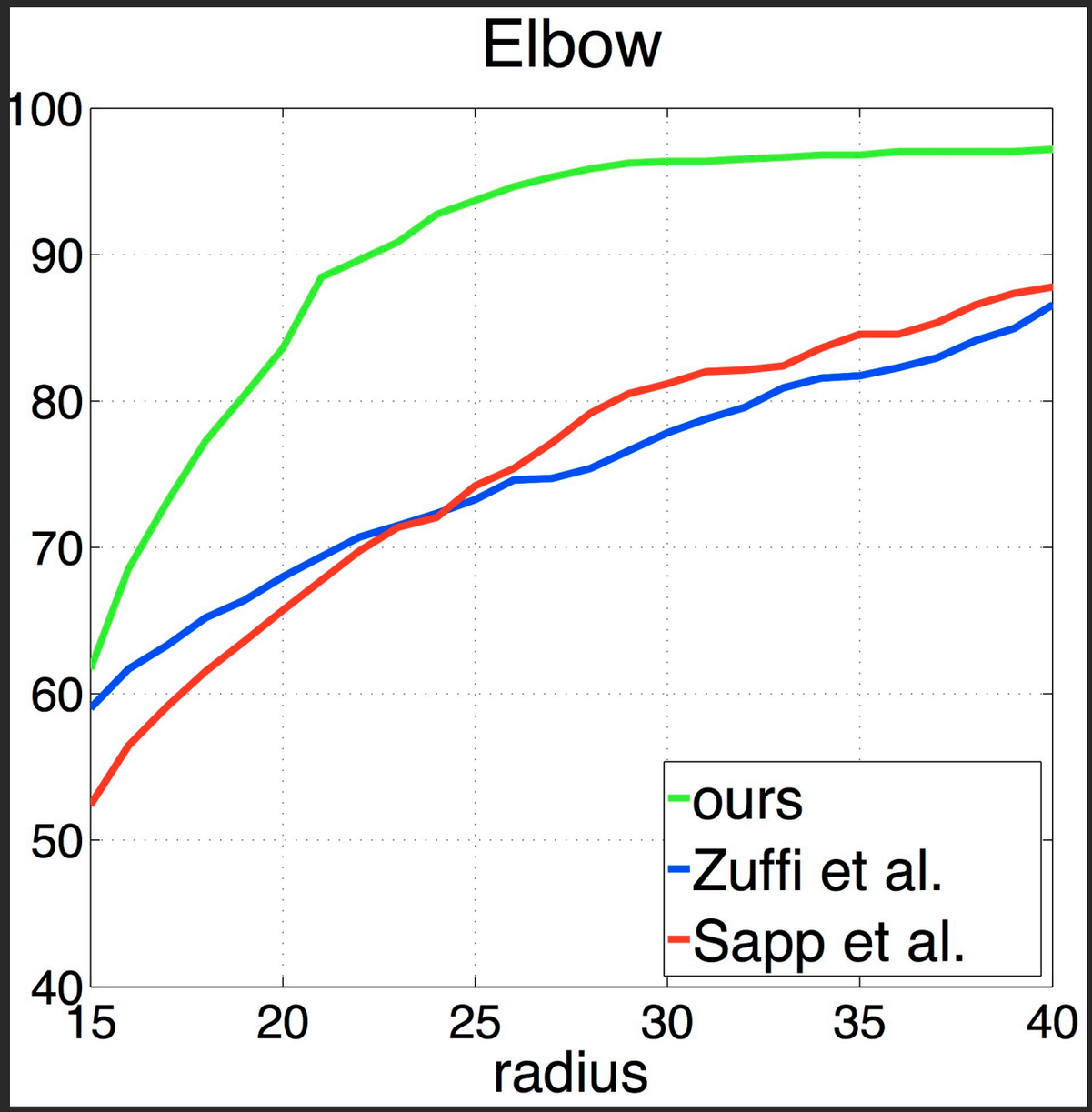


General training set

Video-specific training set

Surprisingly effective

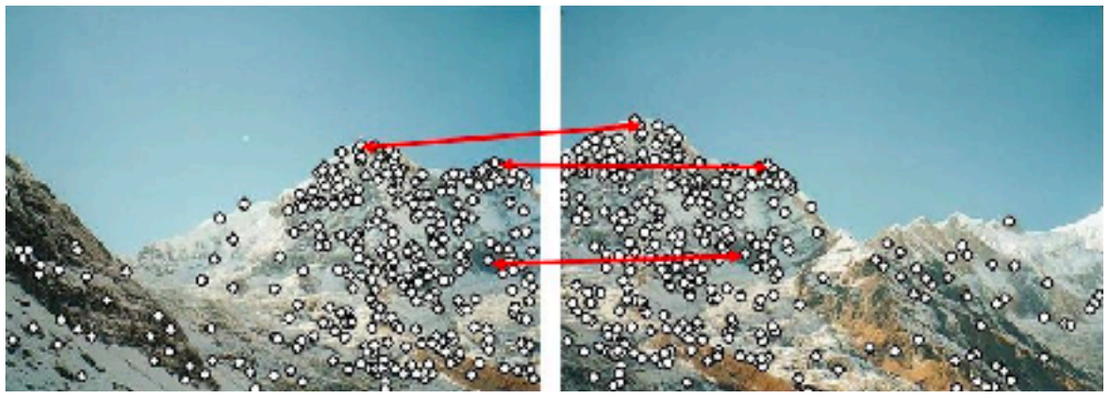
% of correctly-predicted elbows



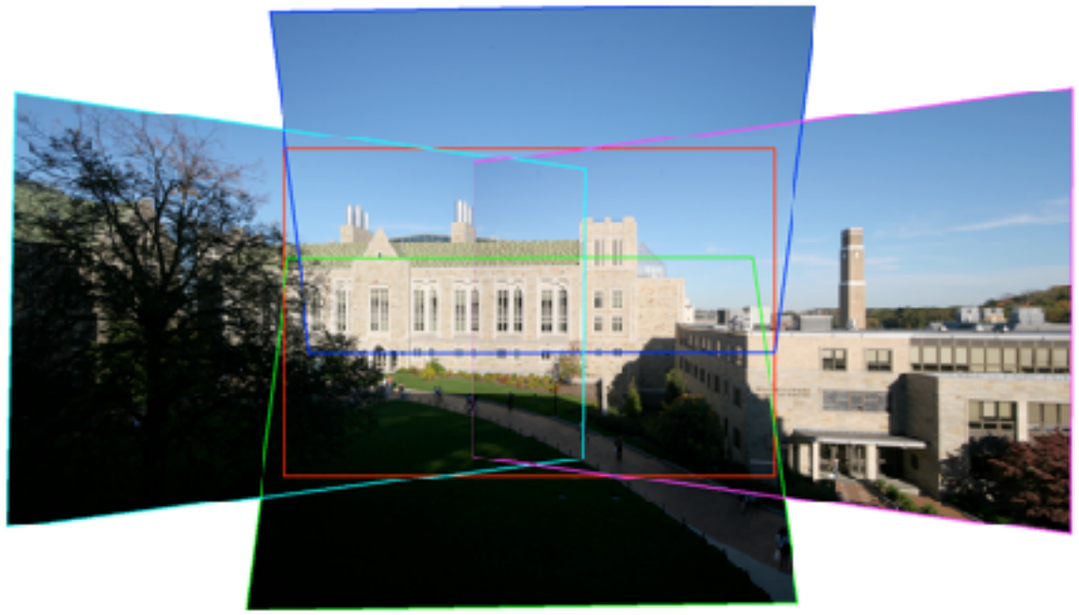
Outline

- Review from last lec
- Transformations (2D/3D)
- Direct methods
- Lucas Kanade

Approaches for image alignmnment

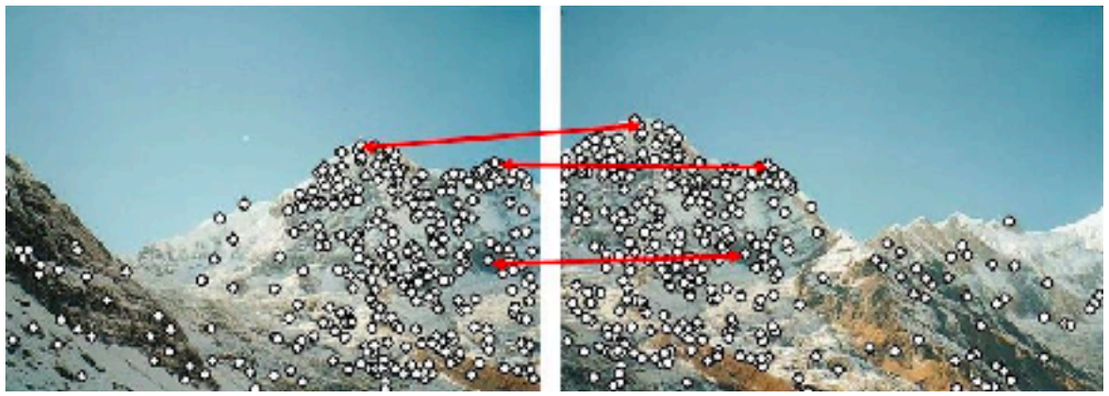


Sparse feature-based alignment

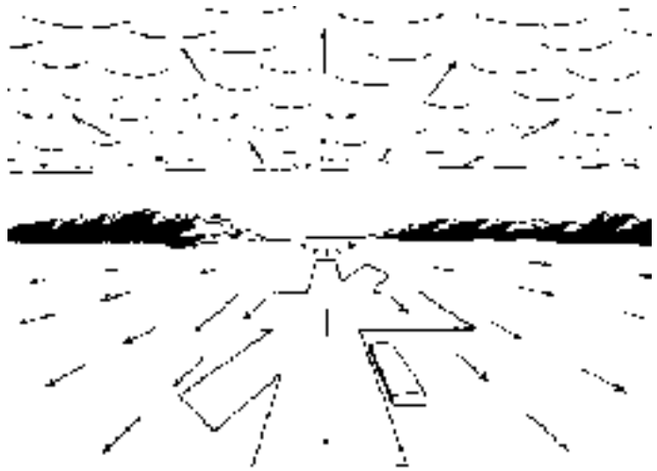


Dense, direct estimation of warp

Approaches for image alignmnment

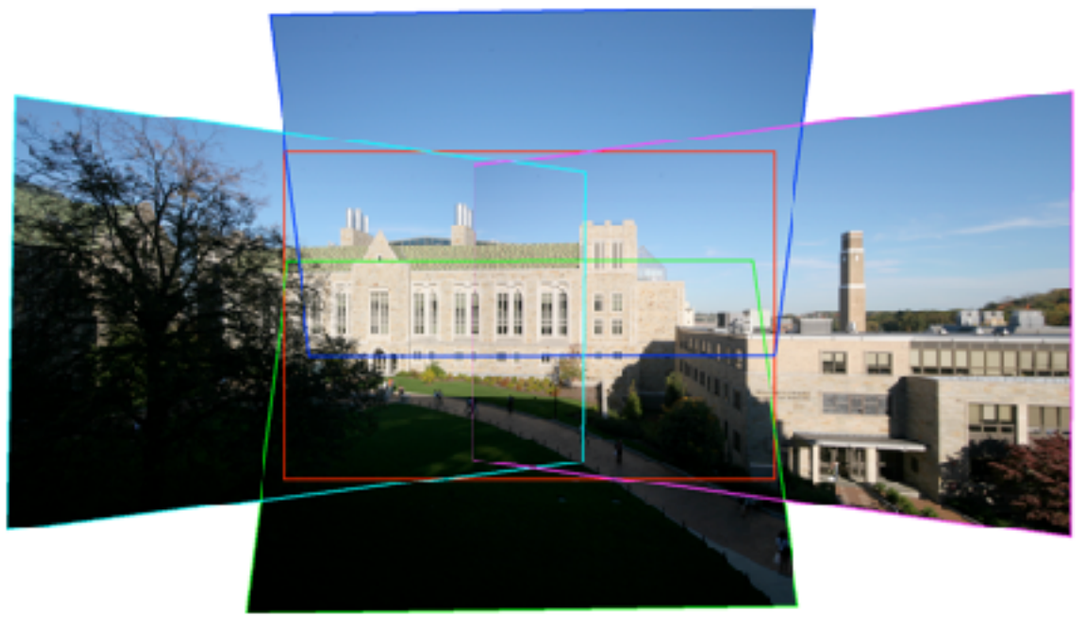


Sparse feature-based alignment



local image transformations (flow fields)

next lecture



global image transformation

Debate at ICCV 1999

In the present context, we define "Direct Methods" as methods for motion and/or shape estimation, which recover the unknown parameters directly from *measurable image quantities* at *each pixel* in the image. This is contrast to the "feature-based methods", which first extract a sparse set of distinct features from each image separately, and then recover and analyze their correspondences in order to determine the motion and shape. Feature-based methods minimize an

Feature Based Methods for Structure and Motion Estimation

P. H. S. Torr[1] and A. Zisserman[2]

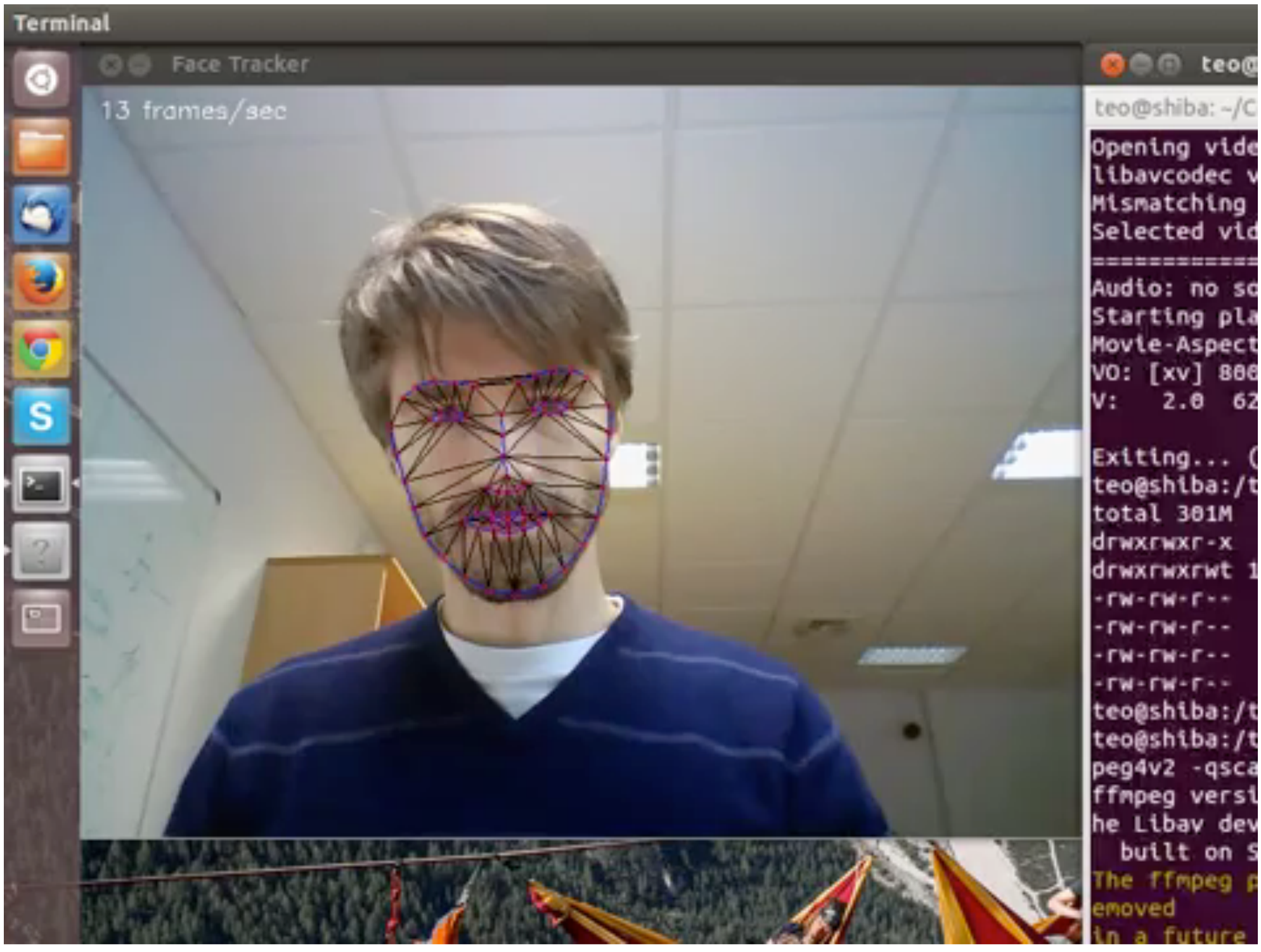
[1] Microsoft Research Ltd, 1 Guildhall St
Cambridge CB2 3NH, UK
philtorr@microsoft.com
[2] Department of Engineering Science, University of Oxford
Oxford, OX1 3PJ, UK
az@robots.ox.ac.uk

All About Direct Methods

M. Irani[1] and P. Anandan[2]

[1] Dept. of Computer Science and Applied Mathematics,
The Weizmann Inst. of Science, Rehovot, Israel.
irani@wisdom.weizmann.ac.il
[2] Microsoft Research, One Microsoft Way,
Redmond, WA 98052, USA.
anandan@microsoft.com

Where we are headed…



Starting point: matching a template to an image

$$E(u,v) = \sum [I(x+u, y+v) - T(x,y)]^2$$

current frame

template
(model)

u,v = hypothesized location of
template in current frame

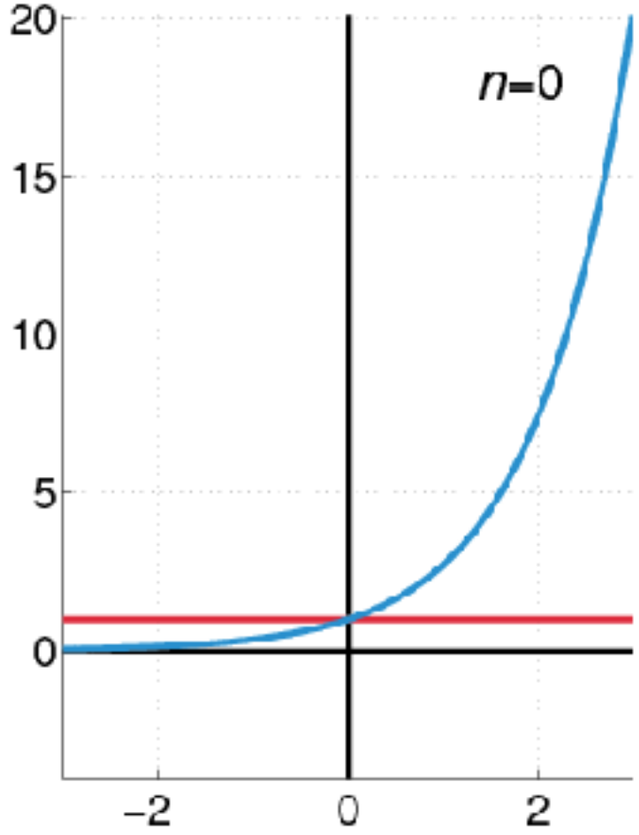
What is the transformation that is being searched over?

What kind of optimization problem is this?

Recall: nonlinear optimization

Apply standard linear optimization tricks after using Taylor series approximation

$$f(x+u) = f(x) + \frac{\partial f(x)}{\partial x}u + \frac{1}{2}\frac{\partial f(x)}{\partial xx}u^2 + \text{Higher Order Terms}$$



For multivariate functions, we make use of first-order gradient vector and second-order Hessian matrix

Recall: nonlinear optimization

$$E(u) = \sum_x [I(x+u) - T(x)]^2$$

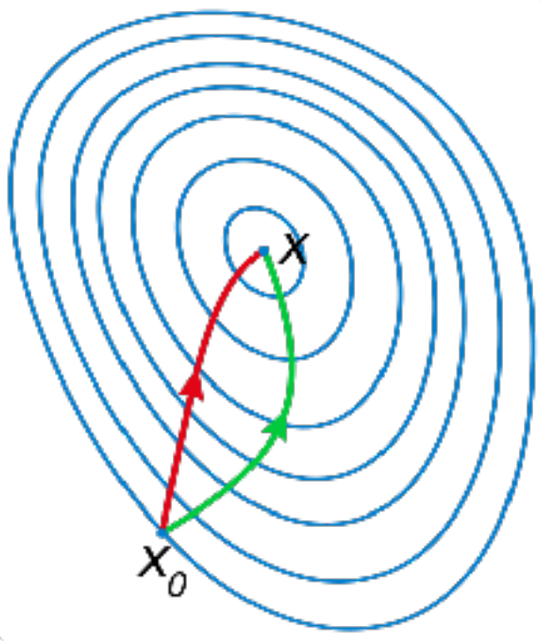
1. First order method (gradient descent)

$$u := u - \alpha g$$

2. Second order method (Newton's method)

https://en.wikipedia.org/wiki/Newton%27s_method_in_optimization

$$u := u - H^{-1} g$$



Recall: nonlinear least squares

$$E(u) = \frac{1}{2}||f(u)||^2$$

1. First order method (gradient descent)

$$u := u - \alpha g$$

2. Second order method (Newton's method)

https://en.wikipedia.org/wiki/Newton's_method_in_optimization

$$u := u - H^{-1}g$$

1.5 Nonlinear least squares (Gauss-Newton approximation)

https://en.wikipedia.org/wiki/Gauss–Newton_algorithm

i. Perform first-order Taylor series expansion of terms inside squared error Δu

ii. Solve quadratic error for Δu

(a) Take derivative of error wrt Δu and set equal to 0

(b) It turns out that we'll compute an "easy to optimize" (PSD) hessian: $H \approx gg^T$

Recall: nonlinear least squares

$$E(u) = \frac{1}{2}||f(u)||^2$$

1. First order method (gradient descent)

$$u := u - \alpha \mathbf{J}(\mathbf{u})^T f(u), \quad J_{ij}(u) = \frac{\partial f_i(u)}{\partial u_j}$$

1.5 Nonlinear least squares (Gauss-Newton approximation)

https://en.wikipedia.org/wiki/Gauss–Newton_algorithm

$$u := u - \left(\mathbf{J}(u)^T \mathbf{J}(u)\right)^T \mathbf{J}(\mathbf{u})^T f(u)$$

Outline

- 2D transformations
- Direct methods
- Lucas Kanade

Lucas Kanade alignment

$$E(u,v) = \sum [I(x+u,y+v) - T(x,y)]^2$$

$$\approx \sum [I(x,y) + uI_x(x,y) + vI_y(x,y) - T(x,y)]^2 \quad \text{First order approx}$$

$$= \sum [uI_x(x,y) + vI_y(x,y) + D(x,y)]^2$$

Take partial derivs and set to zero

Form matrix equation

Lucas Kanade alignment

$$E(u,v) = \sum [I(x+u,y+v) - T(x,y)]^2$$

$$\sum \begin{bmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = -\sum \begin{bmatrix} I_x D \\ I_y D \end{bmatrix}$$

“Ax=b”

Analogy with interest-point detection

Are corners easier to align?

Lucas Kanade Alignment

initialization

template

Lucas Kanade Alignment

Gauss-Newton step

template

Lucas Kanade Alignment

and *iterate*….

template

Is this guaranteed to get the right answer?

Is this garuanteed to converge to any answer?

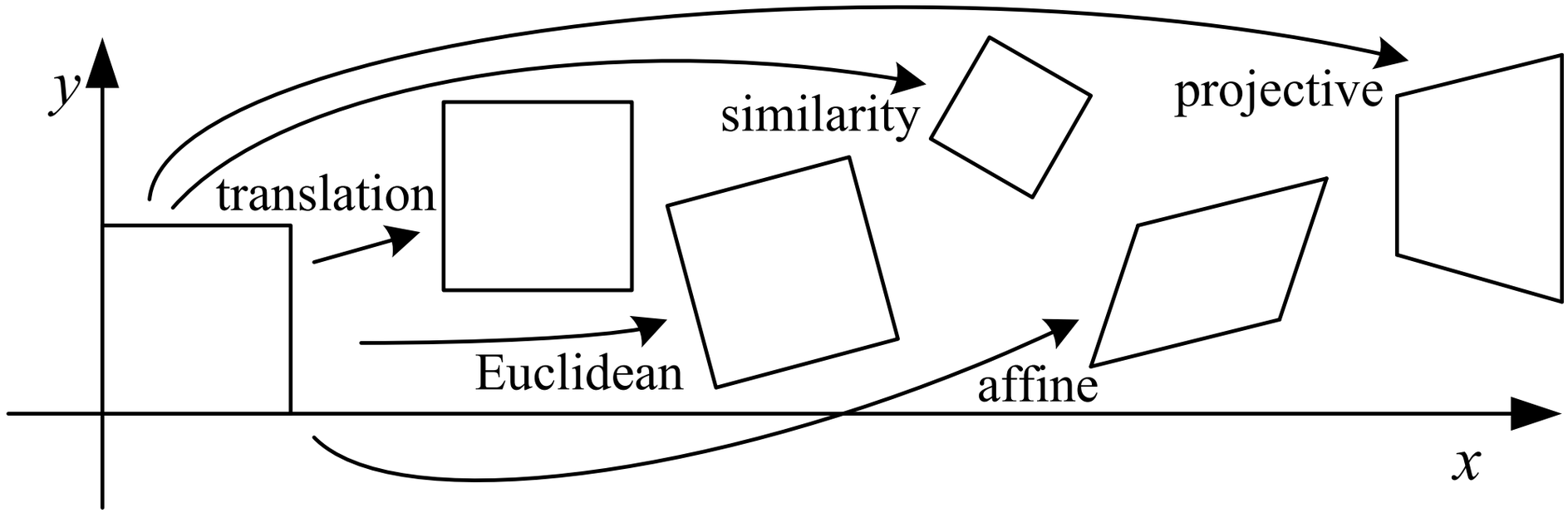
if we are in the *basin of attraction*, convergence tends to be fast (few iterations)

HW3: Tracking by iterative template alignment

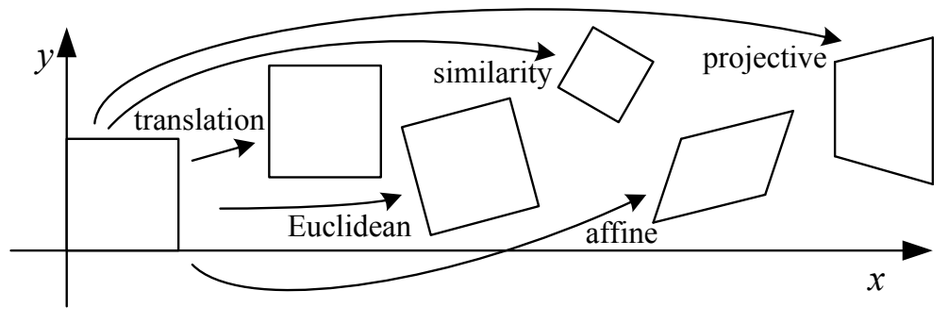
Start with template from first frame and repeat:

1. Align template to new frame with Lucas Kanade
2. Update template with new frame

What about other warps?



Replace [u,v] with a warping function W(x,y)



$$E(u,v) = \sum [I(x+u, y+v) - T(x,y)]^2$$

$$E(p) = \sum [I(W([x,y];p) - T[x,y])^2$$

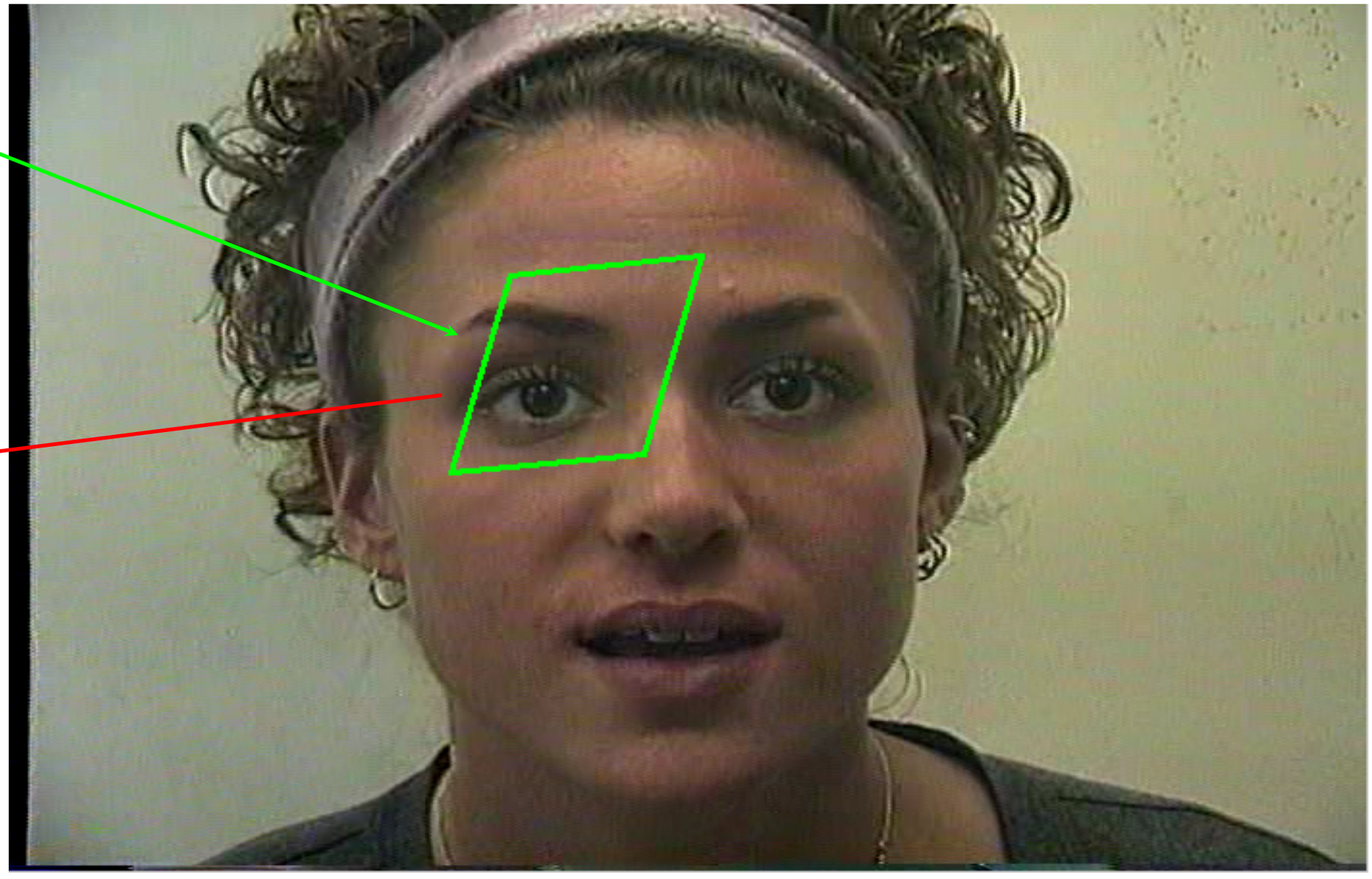
e.g., for affine warps: $W([x,y];p]) = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Want to: Minimize the Error

- Warp image to get $I(\mathbf{W}(\mathbf{x};\mathbf{p}))$ compute $[T(\mathbf{x}) - I(\mathbf{W}(\mathbf{x};\mathbf{p}))]$

Template, $T(\mathbf{x})$

Warped, $I(\mathbf{W}(\mathbf{x};\mathbf{p}))$



Nonlinear least squares

$$E(\mathbf{p}) = \sum_{\mathbf{x}} [I(\mathbf{W}(\mathbf{x};\mathbf{p})) - T(\mathbf{x})]^2$$

Apply taylor-series expansion to vector-valued function $\mathbf{W}$

$$\mathbf{W}(\mathbf{x};\mathbf{p}) = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$W_x = p_1 x + p_2 y + p_3$$

$$W_y = p_4 x + p_5 y + p_6$$

Make use of chain rule to compute $\frac{\partial I}{\partial p_i}$

$$\frac{\partial I}{\partial p_i} = \frac{\partial I}{\partial W_x}\frac{\partial W_x}{\partial p_i} + \frac{\partial I}{\partial W_y}\frac{\partial W_y}{\partial p_i}$$

Taylor expansion of warped image

single-parameter warp (e.g., rotation)

$$I(\mathbf{x};\tilde{p}+\Delta p) \approx I(\mathbf{x};\tilde{p}) + \left[\frac{\partial I(\mathbf{x};\tilde{p})}{\partial x} \quad \frac{\partial I(\mathbf{x};\tilde{p})}{\partial y}\right] \left[\begin{array}{c}\frac{\partial W_x(\mathbf{x},p)}{\partial p} \\ \frac{\partial W_y(\mathbf{x},p)}{\partial p}\end{array}\right]_{\tilde{p}} \Delta p$$

current warped image

current warped image gradient

jacobian of warp

parameter update

Taylor expansion of warped image

single-parameter warp (e.g., rotation)

$$I(\mathbf{x};\tilde{p}+\Delta p) \approx I(\mathbf{x};\tilde{p}) + \left[\frac{\partial I(\mathbf{x};\tilde{p})}{\partial x} \quad \frac{\partial I(\mathbf{x};\tilde{p})}{\partial y}\right] \left[\begin{array}{c}\frac{\partial W_x(\mathbf{x},p)}{\partial p} \\ \frac{\partial W_y(\mathbf{x},p)}{\partial p}\end{array}\right]_{\tilde{p}} \Delta p$$

current warped image | current warped image gradient | jacobian of warp | parameter update

$$I(\mathbf{x};\tilde{\mathbf{p}}+\mathbf{\Delta p}) \approx I(\mathbf{x};\tilde{\mathbf{p}}) + \left[\frac{\partial I(\mathbf{x};\tilde{\mathbf{p}})}{\partial x} \quad \frac{\partial I(\mathbf{x};\tilde{\mathbf{p}})}{\partial y}\right] \left[\begin{array}{c}\frac{\partial W_x(\mathbf{x},\mathbf{p})}{\partial \mathbf{p}} \\ \frac{\partial W_y(\mathbf{x},\mathbf{p})}{\partial \mathbf{p}}\end{array}\right]_{\tilde{\mathbf{p}}} \mathbf{\Delta p}$$

current warped image | current warped image gradient | jacobian matrix | parameter update vector

Example: jacobian of affine warp

affine warp function (6 parameters)

$$W([x,y];P) = \begin{pmatrix} 1+p_1 & p_3 & p_5 \\ p_2 & 1+p_4 & p_6 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\frac{\partial W}{\partial P} = \frac{\partial \begin{bmatrix} x + xP_1 + yP_3 + P_5 \\ xP_2 + y + yP_4 + P_6 \end{bmatrix}}{\partial P}$$

$$= \begin{bmatrix} x & 0 & y & 0 & 1 & 0 \\ 0 & x & 0 & y & 0 & 1 \end{bmatrix}$$

Above affine parameterization is better conditioned for optimization because all-zero parameters default to the identity transformation

Back to the big-picture

$$\sum_{\mathbf{x}} [\, I(\mathbf{W}(\mathbf{x};\mathbf{p}+\Delta\mathbf{p})) - T(\mathbf{x})\,]^2$$

$$\approx \sum_{\mathbf{x}} \left[I(\mathbf{W}(\mathbf{x};\mathbf{p})) + \boldsymbol{\nabla} I \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \Delta\mathbf{p} - T(\mathbf{x}) \right]^2$$

Gradient Descent Solution

Least squares problem: Minimize to solve for $\Delta\mathbf{p}$

$$\sum_{\mathbf{x}} \left[I(\mathbf{W}(\mathbf{x};\mathbf{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \Delta\mathbf{p} - T(\mathbf{x}) \right]^2$$

Solution,

$$\Delta\mathbf{p} = \sum_{\mathbf{x}} H^{-1} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \right]^{\mathrm{T}} [T(\mathbf{x}) - I(\mathbf{W}(\mathbf{x};\mathbf{p}))]$$

Error Image

Gradient

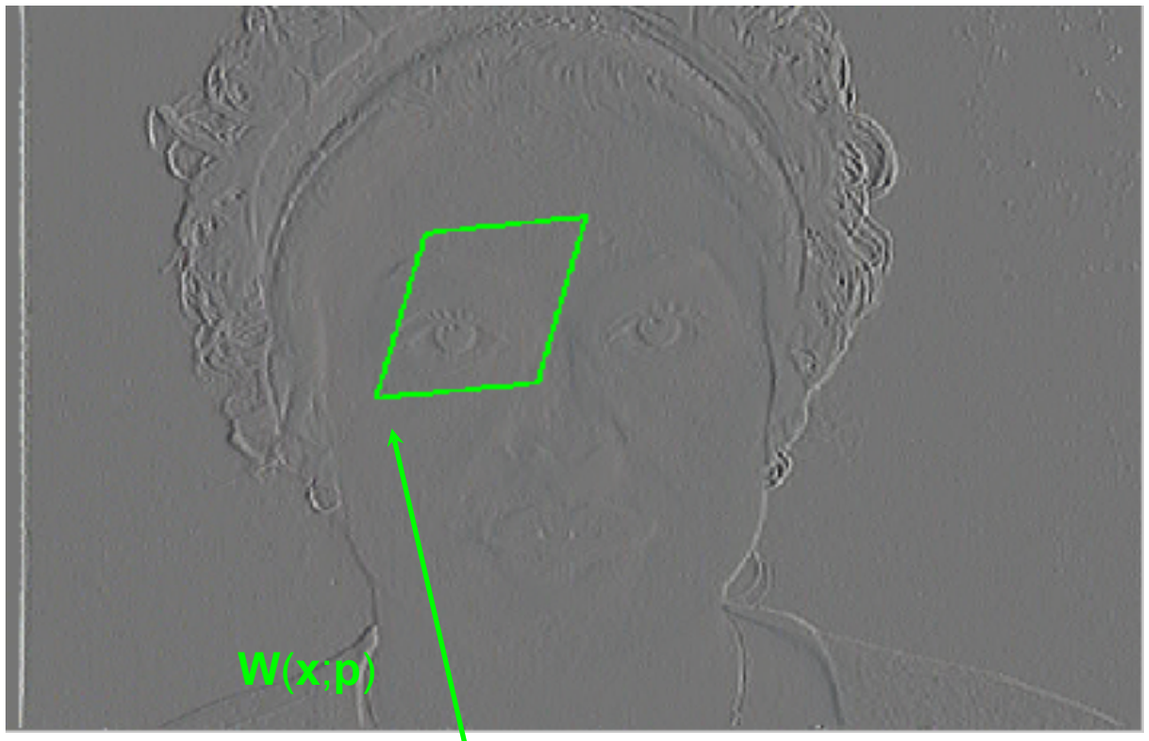
$$H = \sum_{\mathbf{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \right]^{\mathrm{T}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \right]$$

Approx hessian

Jacobian

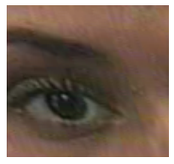
Gradient Images

- Compute image gradient ∇I



∇I_x

$I(\mathbf{W}(\mathbf{x};\mathbf{p}))$



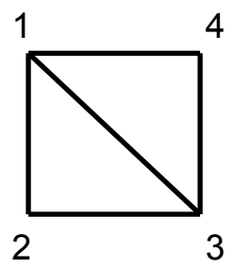
∇I_y

Jacobian

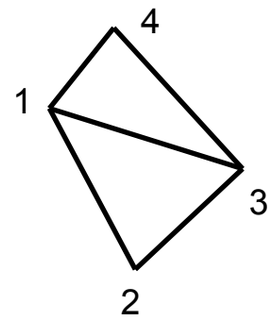
- Compute Jacobian $\frac{\partial \mathbf{W}}{\partial \mathbf{p}}$

  Mesh parameterization

Template, $T(\mathbf{x})$



Warp, $\mathbf{W}(\mathbf{x};\mathbf{p})$



Image, $I(\mathbf{x})$

Image coordinates

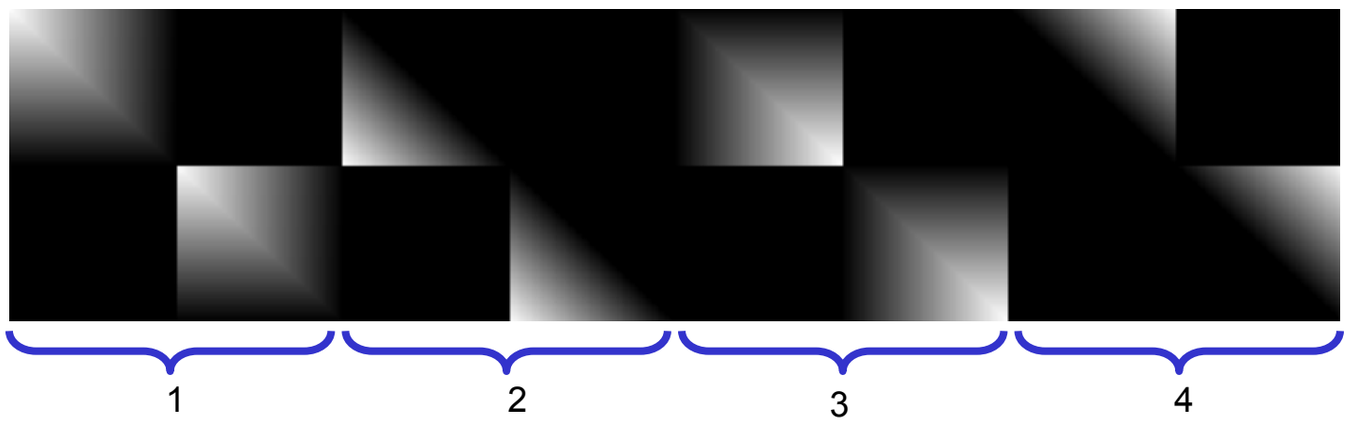
$\mathbf{x} = (x, y)^T$

Warp parameters,

$\mathbf{p} = (p_1, p_2, \ldots, p_n)^T = (dx_1, dy_1, \ldots, dx_n, dy_n)^T$

$$\mathbf{W} = (W_x(x,y), W_y(x,y))$$

$$\frac{\partial \mathbf{W}}{\partial \mathbf{p}} = \begin{pmatrix} \frac{\partial W_x}{\partial p_1} & \frac{\partial W_x}{\partial p_2} & \cdots & \frac{\partial W_x}{\partial p_n} \\ \frac{\partial W_y}{\partial p_1} & \frac{\partial W_y}{\partial p_2} & \cdots & \frac{\partial W_y}{\partial p_n} \end{pmatrix} =$$

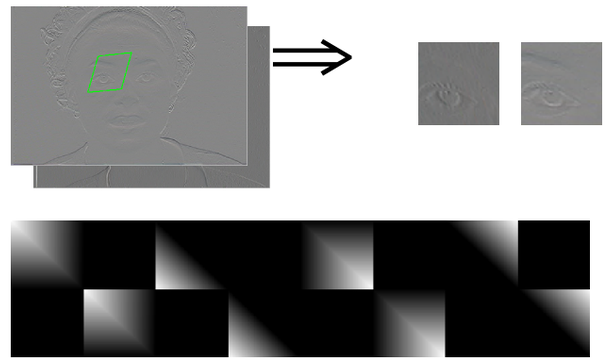


Lucas-Kanade Algorithm

1. Warp I with $\mathbf{W}(\mathbf{x};\mathbf{p}) \Rightarrow I(\mathbf{W}(\mathbf{x};\mathbf{p}))$

1. Compute error image $T(\mathbf{x}) - I(\mathbf{W}(\mathbf{x};\mathbf{p}))$

1. Warp gradient of I to compute ∇I



1. Evaluate Jacobian $\frac{\partial \mathbf{W}}{\partial \mathbf{p}}$

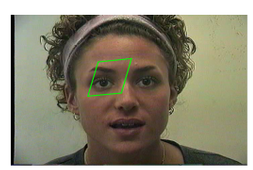
1. **Compute Hessian**

$$H = \sum_{\mathbf{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \mathbf{p}}\right]^{\mathrm{T}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \mathbf{p}}\right]$$

1. Compute $\Delta \mathbf{p}$

$$\Delta \mathbf{p} = \sum_{\mathbf{x}} H^{-1} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \mathbf{p}}\right]^{\mathrm{T}} [T(\mathbf{x}) - I(\mathbf{W}(\mathbf{x};\mathbf{p}))]$$

1. Update parameters $\mathbf{p} \leftarrow \mathbf{p} + \Delta \mathbf{p}$



Fast Gradient Descent?

$$\Delta \mathbf{p} = \sum_{\mathbf{x}} H^{-1} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \right]^{\mathbf{T}} [T(\mathbf{x}) - I(\mathbf{W}(\mathbf{x};\mathbf{p}))]$$

$$H = \sum_{\mathbf{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \right]^{\mathbf{T}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \right]$$

- To reduce Hessian computation:
  1. Make Jacobian simple (*or constant*)
  2. Avoid computing gradients on *I*

Fantastic reference

Lucas-Kanade 20 Years On: A Unifying Framework

SIMON BAKER AND IAIN MATTHEWS
The Robotics Institute, Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh, PA 15213, USA
simonb@cs.cmu.edu
iainm@cs.cmu.edu

IJCV 2004

Overview

Additive warp: $\sum_{\mathbf{x}} [I(\mathbf{W}(\mathbf{x}; \mathbf{p} + \Delta\mathbf{p})) - T(\mathbf{x})]^2$ $\mathbf{p} \leftarrow \mathbf{p} + \Delta\mathbf{p}$.

Compositional warps: $\sum_{\mathbf{x}} [I(\mathbf{W}(\mathbf{W}(\mathbf{x}; \Delta\mathbf{p}); \mathbf{p})) - T(\mathbf{x})]^2$ $\mathbf{W}(\mathbf{x}; \mathbf{p}) \leftarrow \mathbf{W}(\mathbf{x}; \mathbf{p}) \circ \mathbf{W}(\mathbf{x}; \Delta\mathbf{p})$,

$$\mathbf{W}(\mathbf{x}; \mathbf{p}) = \begin{pmatrix} (1+p_1)\cdot x & + & p_3 \cdot y & + & p_5 \\ p_2 \cdot x & + & (1+p_4)\cdot y & + & p_6 \end{pmatrix}$$

$$\mathbf{W}(\mathbf{x}; \mathbf{p}) \circ \mathbf{W}(\mathbf{x}; \Delta\mathbf{p}) = \begin{pmatrix} (1+p_1)\cdot((1+\Delta p_1)\cdot x + \Delta p_3 \cdot y + \Delta p_5) \\ \quad + p_3 \cdot (\Delta p_2 \cdot x + (1+\Delta p_4)\cdot y + \Delta p_6) \\ \quad + p_5 \\ p_2 \cdot ((1+\Delta p_1)\cdot x + \Delta p_3 \cdot y + \Delta p_5) \\ \quad + (1+p_4)\cdot(\Delta p_2 \cdot x + (1+\Delta p_4)\cdot y \\ \quad + \Delta p_6) + p_6 \end{pmatrix}$$

Work out Taylor expansion; it turns out Jacobian is evaluated at $\mathbf{p} = 0$, which means it can be precomputed

Overview

Additive warp: $\sum_{\mathbf{x}} [I(\mathbf{W}(\mathbf{x};\mathbf{p}+\Delta\mathbf{p})) - T(\mathbf{x})]^2$ $\quad\quad$ $\mathbf{p} \leftarrow \mathbf{p} + \Delta\mathbf{p}$.

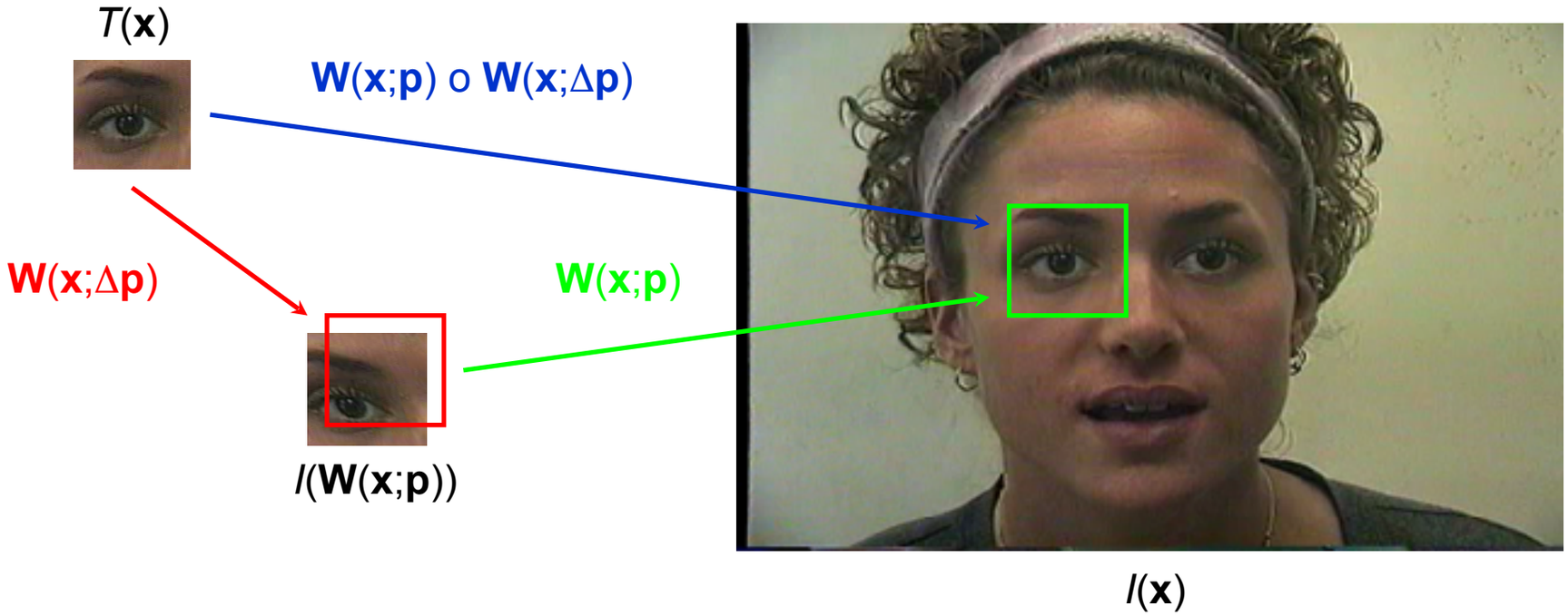
Compositional warp: $\sum_{\mathbf{x}} [I(\mathbf{W}(\mathbf{W}(\mathbf{x};\Delta\mathbf{p});\mathbf{p})) - T(\mathbf{x})]^2$ $\quad\quad$ $\mathbf{W}(\mathbf{x};\mathbf{p}) \leftarrow \mathbf{W}(\mathbf{x};\mathbf{p}) \circ \mathbf{W}(\mathbf{x};\Delta\mathbf{p})$,

Inverse compositional warp: $\sum_{\mathbf{x}} [\, T(\mathbf{W}(\mathbf{x};\Delta\mathbf{p})) - I(\mathbf{W}(\mathbf{x};\mathbf{p}))\,]^2$ $\quad\quad$ $\mathbf{W}(\mathbf{x};\mathbf{p}) \leftarrow \mathbf{W}(\mathbf{x};\mathbf{p}) \circ \mathbf{W}(\mathbf{x};\Delta\mathbf{p})^{-1}$.

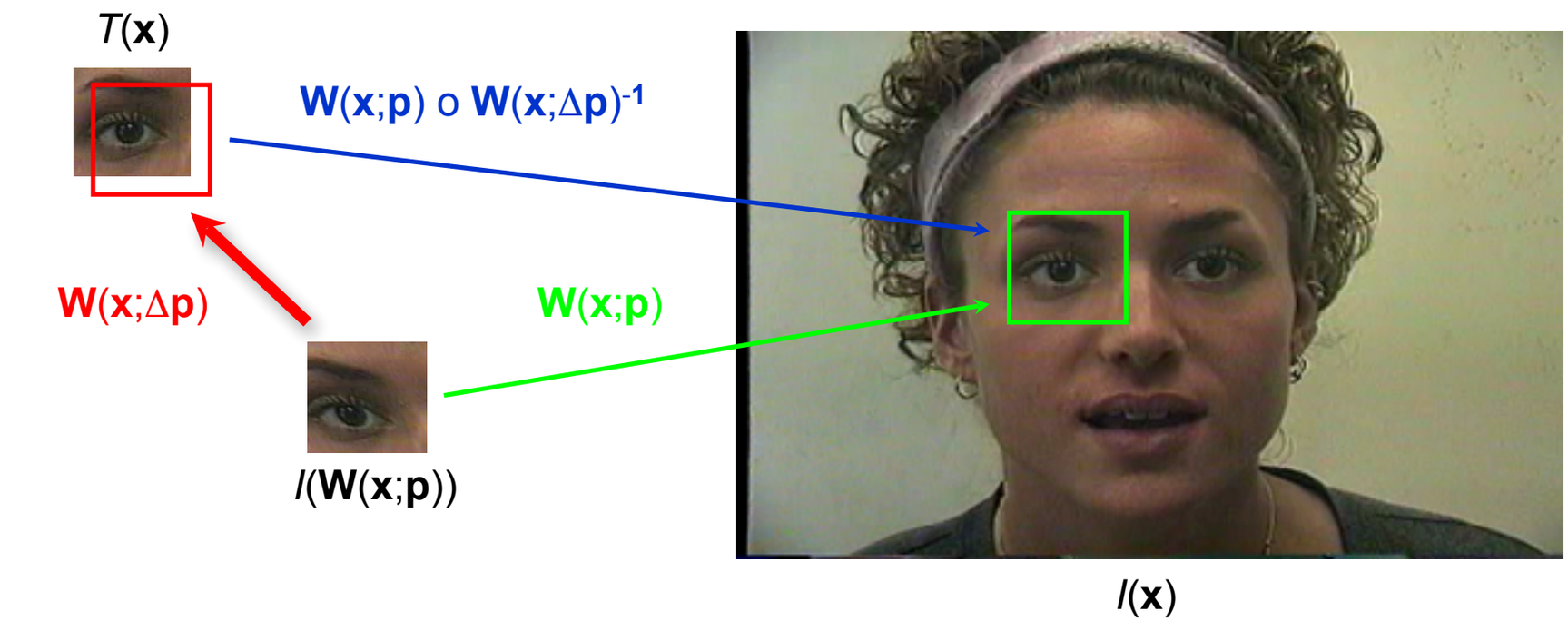
Work out Taylor expansion; both Jacobian and Hessian are not a function of current **p** and so can be precomputed

Forward and Inverse Compositional

- Forwards compositional



- Inverse compositional



Inverse Compositional

- Minimise,

$$\sum_{\mathbf{x}} [T(\mathbf{W}(\mathbf{x};\Delta\mathbf{p})) - I(\mathbf{W}(\mathbf{x};\mathbf{p}))]^2 \approx \sum_{\mathbf{x}} \left[T(\mathbf{W}(\mathbf{x};\mathbf{0})) + \nabla T \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \Delta\mathbf{p} - I(\mathbf{W}(\mathbf{x};\mathbf{p})) \right]^2$$

- Solution

$$H = \sum_{\mathbf{x}} \left[\nabla T \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \right]^{\mathrm{T}} \left[\nabla T \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \right]$$

$$\Delta\mathbf{p} = -\sum_{\mathbf{x}} H^{-1} \left[\nabla T \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \right]^{\mathrm{T}} [T(\mathbf{x}) - I(\mathbf{W}(\mathbf{x};\mathbf{p}))]$$

- Update

$$\mathbf{W}(\mathbf{x};\mathbf{p}) \leftarrow \mathbf{W}(\mathbf{x};\mathbf{p}) \circ \mathbf{W}(\mathbf{x};\Delta\mathbf{p})^{-1}$$

Inverse Compositional

- Jacobian is **constant** - *evaluated at (**x**, **0**)*
- Gradient of template is **constant**
- Hessian is **constant**

$$H = \sum_{\mathbf{x}} \left[\nabla T \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \right]^{\mathrm{T}} \left[\nabla T \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \right]$$

$$\Delta \mathbf{p} = -\sum_{\mathbf{x}} H^{-1} \left[\nabla T \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \right]^{\mathrm{T}} [T(\mathbf{x}) - I(\mathbf{W}(\mathbf{x};\mathbf{p}))]$$

- Can pre-compute everything but error image!

Peicewise affine-tracking